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$\mathcal{C}$-representations of Mixed Abelian Groups III

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First, let $p_1 = 2, p_2 = 3, \ldots$ be a listing of the prime numbers in increasing order, and $\langle a_{p_i m}\rangle$ a cyclic group of order $p_i^{e_i(m)} (m = 0, 1, 2, \ldots)$with $e_i(0) = \cdots = e_i(k_1 - 1) = 1 < e_i(k_1) = \cdots = e_i(k_2 - 1) < e_i(k_2) = \cdots$. And let A be the direct sum of groups $A_{p_i} = \oplus_{m=0}^{\infty} \langle a_{p_i m}\rangle$, one for each i $(i \in \mathbf{N})$, i.e., $A = \oplus_{i\in\mathbf{N}} A_{p_i}$. Also define C by the direct sum of rational groups $C_j = \langle \frac{1}{p_i^{k_i}}$ for $k_i \in \mathbf{N_0} = \mathbf{N} \cup \{0\}$ $(\forall i \in \mathbf{N})\rangle$ of type $\mathbf{t}(C_j) = (\gamma_{j1}, \ldots, \gamma_{ji}, \ldots)$, one for each j $(j = 1, \ldots, n)$, i.e., $C = \oplus_{j=1,\ldots,n} C_j$, $C_j \subseteq \mathbf{Q}$. Then any element $\frac{m_j}{n_j}$ of C_j can be written in the form $\frac{m_j}{n_j} = s_j + \sum_{1\le i\le I_j} \frac{r_{ji}}{p_i^{\alpha_{ji}}}$, where $n_j = \prod_{1\le i\le I_j} p_i^{\alpha_{ji}} (0 \le \alpha_{ji} \le \gamma_{ji})$, $m_j, s_j, r_{ji} \in \mathbf{Z}$, $0 \le r_{ji} < p_i^{\alpha_{ji}}, p_i \nmid r_{ji}$ if $r_{ji} \ne 0$.

The aim of our study is to give extensions $B_\chi^{(\alpha)}$ $(\alpha \in \mathbf{N_0})$ of A by C such that $B_\chi^{(\alpha')}$, $B_\chi^{(\alpha'')}$ $(\alpha', \alpha'' \in \mathbf{N_0})$ are not isomorphic, but $\overline{B_{\chi p_l}^{(\alpha')}} = B_\chi^{(\alpha')}/A_{p_l}^*$, $\overline{B_{\chi p_l}^{(\alpha'')}} = B_\chi^{(\alpha'')}/A_{p_l}^*$ are isomorphic and nonsplitting for any p_l, where $A_{p_l}^* = \oplus_{i\ne l, i\in\mathbf{N}} A_{p_i}$. Therefore, we obtain $\mathcal{C}$-representations of $B_\chi^{(\alpha)}$ $(\alpha \in \mathbf{N_0})$ to find out their structures.

Now, we shall construct a mixed group B_χ as mentioned above. For $\chi_{ji}(m) \in \mathbf{N_0} (m = 0, 1, 2, \ldots, k_t, \ldots)$ with $0 \le \chi_{ji}(m) < p_i^{e_i(m)}$ and $\chi_{ji}(m) = 0$ if $m \equiv j \pmod{n}$, the elements
$a_{p_i 0}^{(\chi_{ji})} = (\chi_{ji}(0)a_{p_i 0}, \ldots, \chi_{ji}(k_1{-}1)a_{p_i k_1 - 1}, \chi_{ji}(k_1)p_i a_{p_i k_1}, \ldots, \chi_{ji}(k_2{-}1)p_i a_{p_i k_2 - 1}, \chi_{ji}(k_2)p_i^2 a_{p_i k_2}, \ldots)$,
$a_{p_i k_t}^{(\chi_{ji})} = (0, \ldots, 0, \chi_{ji}(k_t)a_{p_i k_t}, \ldots, \chi_{ji}(k_{t+1}{-}1)a_{p_i k_{t+1} - 1}, \chi_{ji}(k_{t+1})p_i a_{p_i k_{t+1}}, \chi_{ji}(k_{t+1}{+}1)p_i a_{p_i k_{t+1}+1}, \ldots)$
$\in B_{p_i} = \prod_{m=0}^{\infty} \langle a_{p_i m}\rangle$ $(t \in \mathbf{N})$ are of order $p_i^{e_i}$ or infinite order and satisfy $p_i \nmid a_{p_i 0}^{(\chi_{ji})}$ if $p_i \nmid \chi_{ji}(0)$. Also there are elements $x_{p_l i k_t}^{(\chi_{jl})}$ $(t \in \mathbf{N})$ in B_{p_l} such that $p_i x_{p_l i k_1}^{(\chi_{jl})} = a_{p_l 0}^{(\chi_{jl})}$, $p_i x_{p_l i k_{t+1}}^{(\chi_{jl})} = x_{p_l i k_t}^{(\chi_{jl})}$ for $l \ne i$, $l \in \mathbf{N}$. Thus $\prod_{l\in\mathbf{N}} B_{p_l}$ contains unique elements $b_0^{(\chi_j)} = (a_{p_1 0}^{(\chi_{j1})}, \ldots, a_{p_{i-1} 0}^{(\chi_{ji-1})}, a_{p_i 0}^{(\chi_{ji})}, a_{p_{i+1} 0}^{(\chi_{ji+1})}, \ldots)$, $b_{p_i k_t}^{(\chi_j)} = (x_{p_1 i k_t}^{(\chi_{j1})}, \ldots x_{p_{i-1} i k_t}^{(\chi_{ji-1})}, a_{p_i k_t}^{(\chi_{ji})}, x_{p_{i+1} i k_t}^{(\chi_{ji+1})}, \ldots)$ $(i, t \in \mathbf{N}, j = 1, \ldots, n)$ so as to satisfy $p_i b_{p_i k_1}^{(\chi_j)} = b_0^{(\chi_j)} - \sum_{m=0}^{k_1 - 1} \chi_{ji}(m) a_{p_i m}$, $p_i b_{p_i k_{t+1}}^{(\chi_j)} = b_{p_i k_t}^{(\chi_j)} - \sum_{m=k_t}^{k_{t+1}-1} \chi_{ji}(m) a_{p_i m}$, where $\chi_j = (\chi_{ji})_{i\in\mathbf{N}}$. And, in $\prod_{l\in\mathbf{N}} B_{p_l}$, we consider a mixed group $B_\chi = \langle A, b_0^{(\chi_j)}, b_{p_i k_t}^{(\chi_j)}$ for $i, t \in \mathbf{N}, 0 \le t \le \gamma_{ji}, j = 1, \ldots, n\rangle$ generated by adjoining the elements $b_0^{(\chi_j)}, b_{p_i k_t}^{(\chi_j)} (i, t \in \mathbf{N}, j = 1, \ldots, n)$ to A.

Thereafter, we choose $\chi_{ji}(m)$ as follows : (I) for $0 \le m < k_{\alpha+1}$, $\chi_{ji}(m) = 1$ if $m = 0, \ldots, k_1 - 2, k_t, \ldots, k_{t+1} - 2$ $(1 \le t \le \alpha)$ and $m \not\equiv j \pmod{n}$; $\chi_{ji}(m) = 0$ otherwise. (II) for $k_{\alpha+1} \le m$, $\chi_{ji}(m) = 1$ if $m \not\equiv j \pmod{n}$; $\chi_{ji}(m) = 0$ if $m \equiv j \pmod{n}$. Here denote $B_\chi = B_\chi^{(\alpha)}$ with this $\chi = (\chi_{ji})_{i\in\mathbf{N}, j=1,\ldots,n}$, then $\overline{B_{\chi p_l}^{(\alpha)}} = B_\chi^{(\alpha)}/A_{p_l}^*$.

Next, define a p_l-mixed group $B^{(p_l)}$ in terms of generators and defining relations as follows : It is generated by elements $a_{p_l m}^{(l)}$, $b_{j0}^{(l)}$, $b_{j p_i k_t}^{(l)}$ ($m = 0, 1, 2, \ldots;$ $i, t = 1, 2, \ldots;$ $j = 1, \ldots, n$) such that $p_l^{e_l(m)} a_{p_l m}^{(l)} = 0^{(l)}$, $p_l b_{j p_l k_1}^{(l)} = b_{j0}^{(l)} - \sum_{0\le m\le k_1 - 1, m\not\equiv j \pmod{n}} a_{p_l m}^{(l)}$, $p_l b_{j p_l k_{t+1}}^{(l)} = b_{j p_l k_t}^{(l)} - \sum_{k_t \le m \le k_{t+1}-1, m\not\equiv j \pmod{n}} a_{p_l m}^{(l)}$, $p_i b_{j p_i k_1}^{(l)} = b_{j0}^{(l)}$, $p_i b_{j p_i k_{t+1}}^{(l)} = b_{j p_i k_t}^{(l)}$ ($i \ne l$, $i, t = 1, 2, \ldots$).
Then $A^{(p_l)} = \langle a_{p_l m}^{(l)}$ for $m = 0, 1, 2, \ldots\rangle$ is the torsion part of $B^{(p_l)}$. And the correspondence $\sum_{j=1,\ldots,n} \tau_j (s_j + \sum_{1\le i\le I_j} \frac{r_{ji}}{p_i^{\alpha_{ji}}}) \longmapsto \sum_{j=1,\ldots,n} \left\{ s_j (b_{j0}^{(l)} + A^{(p_l)}) + \sum_{1\le i\le I_j} r_{ji} (b_{j p_i k_{\alpha_{ji}}}^{(l)} + A^{(p_l)}) \right\}$

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induces an isomorphism $\kappa^{(p_l)}$ from C onto $B^{(p_l)}/A^{(p_l)}$, where $\mathbf{t}(C_j) = (\infty, \ldots, \infty, \ldots)$ and the coordinate injection $\tau_j : \frac{m_j}{n_j} \longmapsto (0, \ldots, 0, \frac{m_j}{n_j}, 0, \ldots, 0) \in C$. With any elements $a^{(l)}_{p_l m}, b^{(l)}_{j0}, b^{(l)}_{jp_l k_t}$ and $b^{(l)}_{jp_i k_t}$ $(m = 0, 1, \ldots;\ i \neq\ l,\ i, t \in \mathbf{N}, j = 1, \ldots, n)$ of $B^{(p_l)}$, we associate elements $a_{p_l m} + A^*_{p_l}, b^{(\chi_j)}_0 + \sum_{r=0}^{\alpha} \chi_{jl}(k_{r+1} - 1)p_l^r a_{p_l k_{r+1}-1} + A^*_{p_l},\ b^{(\chi_j)}_{p_l k_t} + \sum_{r=t}^{\alpha} \chi_{jl}(k_{r+1} - 1)p_l^{r-t} a_{p_l k_{r+1}-1} + A^*_{p_l} (1 \leq t \leq \alpha),\ b^{(\chi_j)}_{p_l k_t} + A^*_{p_l} (\alpha < t)$ and $b^{(\chi_j)}_{p_i k_t} + \sum_{r=0}^{\alpha} u_{itk_{r+1}-1} p_l^r a_{p_l k_{r+1}-1} + A^*_{p_l}$ of $\overline{B^{(\alpha)}_{\chi p_l}}$, respectively, where $u_{itk_{r+1}-1} \in \mathbf{N_0}, 0 \leq u_{itk_{r+1}-1} < p_l^{e_l(k_{r+1}-1)-r}$, $p_i^t u_{itk_{r+1}-1} \equiv \chi_{jl}(k_{r+1}-1) \pmod{p_l^{e_l(k_{r+1}-1)-r}} (r = 0, 1, \ldots, \alpha)$. This association gives rise to an isomorphism $\rho^{(\alpha)}_{p_l}$ from $B^{(p_l)}$ onto $\overline{B^{(\alpha)}_{\chi p_l}}$ for any p_l.

Since, we consider $\mathcal{C}$-representations of $B^{(\alpha)}_{\chi}$ $(\alpha \in \mathbf{N_0})$. For $u = \sum_{j=1,\ldots,n} \tau_j(s_j + \sum_{1 \leq i \leq I_j} \frac{r_{ji}}{p_i^{\alpha_{ji}}}) \in C$, choose a representative $g^{(\alpha)}_{\chi}(u) = \sum_{j=1,\ldots,n}(s_j b^{(\chi_j)}_0 + \sum_{1 \leq i \leq I_j} r_{ji} b^{(\chi_j)}_{p_i k_{\alpha_{ji}}}) \in C$ of the coset $\kappa^{(\alpha)}_{\chi}(u) = g^{(\alpha)}_{\chi}(u) + A$. Further put $\overline{g^{(\alpha)}_{\chi p_l}}(u) = g^{(\alpha)}_{\chi}(u) + A^*_{p_l}$, then the following holds $(\rho^{(\alpha)}_{p_l})^{-1}\overline{g^{(\alpha)}_{\chi p_l}}(u) = \sum_{j=1,\ldots,n} \lfloor s_j b^{(l)}_{j0} + \sum_{1 \leq i \leq I_j} r_{ji} b^{(l)}_{jp_i k_{\alpha_{ji}}} - \sum_{i=1,\ldots,I_j} \{(1 - \delta_{li}) \sum_{r=0}^{\alpha} r_{ji} u_{i\alpha_{ji}k_{r+1}-1} p_l^r a^{(l)}_{p_l k_{r+1}-1} - \delta_{li} r_{ji} \sum_{r=\alpha_{jl}}^{\alpha} \chi_{jl}(k_{r+1} - 1) p_l^{r-\alpha_{jl}} a^{(l)}_{p_l k_{r+1}-1}\} - s_j \sum_{r=0}^{\alpha} \chi_{jl}(k_{r+1} - 1) p_l^r a^{(l)}_{p_l k_{r+1}-1} \rfloor$, where $\delta_{li} = 1$ if $l = i$; $\delta_{li} = 0$ if $l \neq i$. The function $g^{(p_l)}_{\chi} = (\rho^{(\alpha)}_{p_l})^{-1}\overline{g^{(\alpha)}_{\chi p_l}}$ becomes a representative function from C to $B^{(p_l)}$ relative to $\kappa^{(p_l)}$, which yields the factor set $f^{(p_l)}_{\chi}$ on C to $A^{(p_l)}$ as follows : $f^{(p_l)}_{\chi}(u', u'') = \sum_{j=1,\ldots,n} \sum_{i=1,\ldots,I_j} f^{(p_l)}_{\chi}\big(\tau_j\big(\frac{r_{ji'}}{p_l^{\alpha_{ji'}}}\big), \tau_j\big(\frac{r_{ji''}}{p_l^{\alpha_{ji''}}}\big)\big)$ for $u' = \sum_{j=1,\ldots,n} \tau_j(s'_j + \sum_{1 \leq i \leq I_j} \frac{r_{ji'}}{p_i^{\alpha_{ji'}}}) \in C$ and $u'' = \sum_{j=1,\ldots,n} \tau_j(s''_j + \sum_{1 \leq i \leq I_j} \frac{r_{ji''}}{p_i^{\alpha_{ji''}}}) \in C$. We distinguish two cases. Case I : for $i \neq l,\ i \in \mathbf{N}$ $f^{(p_l)}_{\chi}\big(\tau_j\big(\frac{r_{ji'}}{p_l^{\alpha_{ji'}}}\big), \tau_j\big(\frac{r_{ji''}}{p_l^{\alpha_{ji''}}}\big)\big) = 0^{(l)}$. Case II : for $i = l$ (1) $\alpha_{jl'} \neq \alpha_{jl''}$, i.e., $\alpha_{jl'} > \alpha_{jl''}$ without loss of generality. $f^{(p_l)}_{\chi}\big(\tau_j\big(\frac{r_{jl'}}{p_l^{\alpha_{jl'}}}\big), \tau_j\big(\frac{r_{jl''}}{p_l^{\alpha_{jl''}}}\big)\big) = -\sum_{t=0}^{\alpha_{jl''}-1} \sum_{m=k_t}^{k_{t+1}-1} \chi_{jl}(m) \Delta_{jl} p_l^t a^{(l)}_{p_l m} - \sum_{t=\alpha_{jl''}}^{\alpha_{jl'}-1} \sum_{m=k_t}^{k_{t+1}-1} \chi_{jl}(m) p_l^{t-\alpha_{jl''}} (\Delta_{jl} p_l^{\alpha_{jl''}} - r_{jl''}) a^{(l)}_{p_l m}$, where $\Delta_{jl} = \left[\frac{p_l^{\alpha_{jl'}-\alpha_{jl''}} r_{jl''} + r_{jl'}}{p_l^{\alpha_{jl'}}}\right]$ with Gauss' symbol. (2) $\alpha_{jl} = \alpha_{jl'} = \alpha_{jl''}$. $f^{(p_l)}_{\chi}\big(\tau_j\big(\frac{r_{jl'}}{p_l^{\alpha_{jl'}}}\big), \tau_j\big(\frac{r_{jl''}}{p_l^{\alpha_{jl''}}}\big)\big) = -\sum_{t=0}^{\alpha_{jl}-\beta_{jl}-1} \sum_{m=k_t}^{k_{t+1}-1} \chi_{jl}(m) \Delta_{jl} p_l^t a^{(l)}_{p_l m} - \sum_{t=\alpha_{jl}-\beta_{jl}}^{\alpha_{jl}-1} \sum_{m=k_t}^{k_{t+1}-1} \chi_{jl}(m) p_l^{t-(\alpha_{jl}-\beta_{jl})} (\Delta_{jl} p_l^{\alpha_{jl}-\beta_{jl}} + r'''_{jl}) a^{(l)}_{p_l m}$, where $\Delta_{jl} = \left[\frac{r_{jl'} + r_{jl''}}{p_l^{\alpha_{jl}}}\right]$, $r'''_{jl} = \frac{r_{jl}}{p_l^{\beta_{jl}}}$ if $p_l^{\beta_{jl}} \| r_{jl} = r_{jl'} + r_{jl''} - \Delta_{jl} p_l^{\alpha_{jl}}$.

Next, put $\mathcal{C} = \big(C,\ \big[(B^{(p_l)}, \kappa^{(p_l)})\big]_{l \in \mathbf{N}}\big)$, $B^{(*)} = \prod_{l \in \mathbf{N}} B^{(p_l)}$, $A^{(*)} = \oplus_{l \in \mathbf{N}} A^{(p_l)}$, and let μ_{p_l} be a coordinate injection from $A^{(p_l)}$ into $A^{(*)}$ acting via $a_{p_l} \mapsto (0, \cdots, 0, a_{p_l}, 0, \cdots)$. In $B^{(*)}$, put $g^{(\alpha)+}_{\chi}(u) = \Big(g^{(p_l)}_{\chi}(u)\Big)_{l \in \mathbf{N}} \in \prod_{l \in \mathbf{N}} B^{(p_l)}$, then we construct a mixed group $B^{(\alpha)+}_{\chi}$ generated by adjoining $\Big[g^{(\alpha)+}_{\chi}(u)\Big]_{u \in C}$ to $A^{(*)}$, and write $B^{(\alpha)+}_{\chi} = \Big\langle A^{(*)}, \Big[g^{(\alpha)+}_{\chi}(u)\Big]_{u \in C}\Big\rangle$. Here $B^{(\alpha)+}_{\chi}$ is isomorphic to $B^{(\alpha)}_{\chi}$, and $B^{(\alpha)+}_{\chi} = B\big(\mathcal{C},\ \Big[g^{(p_l)}_{\chi}(u)\Big]_{u \in C, l \in \mathbf{N}}\big)$ is a $\mathcal{C}$-representation of $B^{(\alpha)}_{\chi}$ with respect to representative functions. On the other hand, we construct a group $B^{(\alpha)-}_{\chi}$ as the set of all pairs $\big(u, a^{(*)}\big)_{\chi} \in C \times A^{(*)}$ with the operation $\big(u', a^{(*)\prime}\big)_{\chi} + \big(u'', a^{(*)\prime\prime}\big)_{\chi} = \Big(u' + u'', a^{(*)\prime} + a^{(*)\prime\prime} + f^{(\alpha)-}_{\chi}(u', u'')\Big)_{\chi}$, where $f^{(\alpha)-}_{\chi}$ is a factor set on C to $A^{(*)}$ as follows : $f^{(\alpha)-}_{\chi}(u', u'') = \sum_{l \in \mathbf{N}} \mu_{p_l} f^{(p_l)}_{\chi}(u', u'')$. Here $B^{(\alpha)-}_{\chi}$ is isomorphic to $B^{(\alpha)}_{\chi}$, and $B^{(\alpha)-}_{\chi} = B\big(\mathcal{C},\ \Big[f^{(p_l)}_{\chi}(u', u'')\Big]_{u', u'' \in C, l \in \mathbf{N}}\big)$ is a $\mathcal{C}$-representation of $B^{(\alpha)}_{\chi}$ with respect to factor sets.

Finally, we point out that it is possible to prove that the groups $B^{(\alpha)}_{\chi}$ $(\alpha \in \mathbf{N_0})$ satisfy the conditions as desired by using the above results.

References

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