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$\mathcal{C}$-representations of Mixed Abelian Groups V

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From the studies before, we have obtained the following mixed groups and $\mathcal{C}$-representations of them : $B^{(\alpha)} = \langle a_{p_i m}, b^{(\alpha)}_{jp_i^t} \text{ for } i \in \mathbf{N}, m, t \in \mathbf{N_0}, j = 1, \cdots, n\rangle$ $(\alpha \in \mathbf{N_0})$, where the generators $a_{p_i m}, b^{(\alpha)}_{jp_i^t}$ satisfy $o(a_{p_i m}) = p_i^{e(m)}$ for $e(m) = 2\left[\frac{m}{n}\right] + 1$(an integer $n \geqq 3$) with Gauss' symbol $[\quad]$, $b^{(\alpha)}_{jp_i^0} = b^{(\alpha)}_{j1}, p_i b^{(\alpha)}_{jp_i^{t+1}} - b^{(\alpha)}_{jp_i^t} = -\sum_{m=tn}^{(t+1)n-1} \chi_j^{(\alpha)}(m) a_{p_i m}$. And choose $\chi_j^{(\alpha)}(m)$ $(\alpha, m \in \mathbf{N_0}, j = 1, \cdots, n)$ as follows : (I)$\chi_j^{(\alpha)}(0) = 1$ $(\alpha \in \mathbf{N_0})$ (II)for $m > 0$, $\chi_j^{(0)}(m) = 1$ if $m \not\equiv j \pmod n$; $\chi_j^{(0)}(m) = 0$ if $m \equiv j \pmod n$ (III) (i) for $0 < m < \alpha n$ $(\alpha \in \mathbf{N})$, $\chi_j^{(\alpha)}(m) = 1$ if $m = 1, \cdots, n-3,\quad tn, \cdots, (t+1)n - 3$, $(0 < t < \alpha)$ and $m \not\equiv j \pmod n$; $\chi_j^{(\alpha)}(m) = 0$ if otherwise (ii) for $0 < \alpha n \leqq m$ $(\alpha \in \mathbf{N})$, $\chi_j^{(\alpha)}(m) = \chi_j^{(0)}(m)$.

Also let $\widetilde{B}_{p_l} = \langle \tilde{a}^{[l]}_{p_l m}, \tilde{b}^{[l]}_{jp_i^t} \text{ for } i \in \mathbf{N}, m, t \in \mathbf{N_0}, j = 1, \cdots, n\rangle$ be defined by the defining relations $p_l^{e(m)} \tilde{a}^{[l]}_{p_l m} = \tilde{0}^{[l]}$, $\tilde{b}^{[l]}_{jp_i^0} = \tilde{b}^{[l]}_{j1}$, $p_i \tilde{b}^{[l]}_{jp_i^{t+1}} - \tilde{b}^{[l]}_{jp_i^t} = -\sum_{m=tn}^{(t+1)n-1} \chi_j^{(0)}(m) \delta_{il} \tilde{a}^{[l]}_{p_l m}$ with Kronecker's delta symbol δ_{il}, where $\widetilde{A}_{p_l} = \oplus_{m=0}^{\infty} \langle \tilde{a}^{[l]}_{p_l m}\rangle$ is the torsion part of $\widetilde{B}_{p_l}$. And define $\widetilde{\kappa}_{p_l}$ as an isomorphism from $C = \oplus_{j=1}^{n} \tau_j(\mathbf{Q})$ onto $\widetilde{B}_{p_l}/\widetilde{A}_{p_l}$ induced by the correspondence $u = \sum_{j=1}^{n} \tau_j(s_j + \sum_{i=1}^{I_j} \frac{r_{ji}}{p_i^{\alpha_{ji}}}) \longmapsto \sum_{j=1}^{n} \left\{ s_j(\tilde{b}^{[l]}_{j1} + \widetilde{A}_{p_l}) + \sum_{i=1}^{I_j} r_{ji}(\tilde{b}^{[l]}_{jp_i^{\alpha_{ji}}} + \widetilde{A}_{p_l}) \right\}$, where the coordinate injection $\tau_j : \frac{m_j}{n_j} \longmapsto (0, \cdots, 0, \frac{m_j}{n_j}, 0, \cdots, 0) \in C = \mathbf{Q} \oplus \cdots \oplus \mathbf{Q}$, $\frac{m_j}{n_j} = s_j + \sum_{i=1}^{I_j} \frac{r_{ji}}{p_i^{\alpha_{ji}}}$ with $n_j = \prod_{i=1}^{I_j} p_i^{\alpha_{ji}} (\alpha_{ji} \geqq 0)$, $m_j, s_j, r_{ji} \in \mathbf{Z}$, $0 \leqq r_{ji} < p_i^{\alpha_{ji}}, p_i \nmid r_{ji}$ if $r_{ji} \neq 0$.

Then, $\mathcal{C}$-representations of $B^{(\alpha)}$ $(\alpha \in \mathbf{N_0})$ with respect to representative functions are represented by $B^{(\alpha)+} = B\big(\mathcal{C}, \left[\tilde{g}^{(\alpha)}_{p_l}(u)\right]_{u \in C, l \in \mathbf{N}}\big)$ with $\mathcal{C} = \big(C, \left[(\widetilde{B}_{p_l}, \widetilde{\kappa}_{p_l})\right]_{l \in \mathbf{N}}\big)$ and $\tilde{g}^{(\alpha)}_{p_l}(u) = \tilde{g}^{(0)}_{p_l}(u) - \tilde{h}^{(\alpha)}_{p_l}(u)$. Here $\tilde{g}^{(0)}_{p_l}, \tilde{h}^{(\alpha)}_{p_l}$ are given as follows : $\tilde{g}^{(0)}_{p_l}(u) = \sum_{j=1}^{n}(s_j \tilde{b}^{[l]}_{j1} + \sum_{i=1}^{I_j} r_{ji} \tilde{b}^{[l]}_{jp_i^{\alpha_{ji}}})$,

$$\tilde{h}^{(\alpha)}_{p_l}(u) = \sum_{j=1}^{n} \lfloor \delta_>(\alpha, 0) \sum_{s=1}^{\alpha} \sum_{m=sn-2}^{sn-1} \{s_j \chi_j^{(0)}(m) + \sum_{i=1}^{I_j}(1 - \delta_{il}) r_{ji} r^{(s)}_{jp_i^{\alpha_{ji}}}(m)\} p_l^{s-1} \tilde{a}^{[l]}_{p_l m}$$
$$+ \sum_{i=1}^{I_j} \delta_>(\alpha, \alpha_{ji}) \sum_{s=\alpha_{ji}+1}^{\alpha} \sum_{m=sn-2}^{sn-1} \delta_{il} r_{ji} \chi_j^{(0)}(m) p_l^{s-\alpha_{ji}-1} \tilde{a}^{[l]}_{p_l m} \rfloor,$$

where $\delta_>(\alpha, t) = 1$ if $\alpha > t$; $\delta_>(\alpha, t) = 0$ if $\alpha \leqq t$, and for $l \neq i \in \mathbf{N}, m = (s-1)n, \cdots, sn-1 (s = 1, \cdots, \alpha), r^{(s)}_{jp_i^t}(m) \in \mathbf{Z}$ such that $p_i^t r^{(s)}_{jp_i^t}(m) \equiv \chi_j^{(0)}(m) \pmod{p_l^s}$.

On the other hand, $\mathcal{C}$-representations of $B^{(\alpha)}$ $(\alpha \in \mathbf{N_0})$ with respect to factor sets are represented by $B^{(\alpha)-} = B\big(\mathcal{C}, \left[\tilde{f}^{(\alpha)}_{p_l}(u', u'')\right]_{u', u'' \in C, l \in \mathbf{N}}\big)$ with $\tilde{f}^{(\alpha)}_{p_l}(u', u'') = \tilde{f}^{(0)}_{p_l}(u', u'') - \big(\tilde{h}^{(\alpha)}_{p_l}(u') + \tilde{h}^{(\alpha)}_{p_l}(u'') - \tilde{h}^{(\alpha)}_{p_l}(u' + u'')\big)$. Here, for $u' = \sum_{j=1}^{n} \tau_j(s'_j + \sum_{i=1}^{I_j} \frac{r_{ji}'}{p_i^{\alpha_{ji}'}})$, $u'' = \sum_{j=1}^{n} \tau_j(s''_j + \sum_{i=1}^{I_j} \frac{r_{ji}''}{p_i^{\alpha_{ji}''}}) \in C$, the following holds

$$\tilde{f}^{(\alpha)}_{p_l}(u', u'') = \sum_{j=1}^{n} \sum_{i=1}^{I_j} \{-\delta_>(\alpha_{ji}, 0) \sum_{s=0}^{\alpha_{ji}-1} \sum_{m=sn}^{(s+1)n-1} \chi_j^{(\alpha)}(m) \Delta_{ji} p_i^s \delta_{il} \tilde{a}^{[l]}_{p_l m}$$
$$- \delta_>(\beta_{ji}, 0) \sum_{s=\alpha_{ji}-\beta_{ji}}^{\alpha_{ji}-1} \sum_{m=sn}^{(s+1)n-1} \chi_j^{(\alpha)}(m) r'''_{ji} p_i^{s-(\alpha_{ji}-\beta_{ji})} \delta_{il} \tilde{a}^{[l]}_{p_l m}$$
$$+ \delta_>(\alpha_{ji}, \alpha^-_{ji}) \sum_{s=\alpha^-_{ji}}^{\alpha_{ji}-1} \sum_{m=sn}^{(s+1)n-1} \chi_j^{(\alpha)}(m) r^-_{ji} p_i^{s-\alpha^-_{ji}} \delta_{il} \tilde{a}^{[l]}_{p_l m}\},$$

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where $\Delta_{ji} = \left[\frac{r_{ji}+r_{ji}^{-}p_i^{\alpha_{ji}-\alpha_{ji}^{-}}}{p_i^{\alpha_{ji}}}\right]$, $\quad r_{ji}''' = \frac{\tilde{r}_{ji}}{p_i^{\beta_{ji}}}$ if $p_i^{\beta_{ji}} \| \tilde{r}_{ji} = r_{ji} + r_{ji}^{-}p_i^{\alpha_{ji}-\alpha_{ji}^{-}} - \Delta_{ji}p_i^{\alpha_{ji}}$ with $\alpha_{ji} = \max\{\alpha_{ji}', \alpha_{ji}''\}$, $\alpha_{ji}^{-} = \min\{\alpha_{ji}', \alpha_{ji}''\}$, $r_{ji} = \delta_{\geqq}(\alpha_{ji}', \alpha_{ji}'')r_{ji}' + \delta_{\leqq}(\alpha_{ji}', \alpha_{ji}'')r_{ji}''$, $r_{ji}^{-} = \delta_{<}(\alpha_{ji}', \alpha_{ji}'')r_{ji}' + \delta_{>}(\alpha_{ji}', \alpha_{ji}'')r_{ji}''$. [$\delta_{\geqq}(\alpha, t)$ means that $\delta_{\geqq}(\alpha, t) = 1$ if $\alpha \geqq t$; $\delta_{\geqq}(\alpha, t) = 0$ if $\alpha < t$.]

Hence, we find out that $\tilde{h}_{p_l}^{(\alpha)}(u) \neq \tilde{0}^{[l]}$ for fixed $0 \neq u \in C$ and any p_l, but $\left[\tilde{f}_{p_l}^{(\alpha)}(u', u'')\right]_{u', u'' \in C, l \in \mathbf{N}}$ satisfies the finiteness condition.

Thereafter the aim of our study is to apply the above results in order to investigate the structures of subgroups and p-basic subgroups of $B^{(\alpha)}$ ($\alpha \in \mathbf{N_0}$).

Now put $C' = \oplus_{j=1}^{n}\tau_j(\mathbf{Q}')$, where $\mathbf{Q}' = \{\frac{m_j}{n_j} \mid n_j = \prod_{i=1}^{I_j} p_i^{\alpha_{ji}} (0 \leqq \alpha_{ji} \leqq 2t_0,\ t_0 > \alpha)\}$. And we shall construct a subgroup $B^{(\alpha)\prime}$ of $B^{(\alpha)}$ for any $\alpha \in \mathbf{N_0}$ as follows :

$B^{(\alpha)\prime} = \langle a_{p_i m}, b_{jp_i^t}^{(\alpha)}$ for $i \in \mathbf{N}, t = 0, \cdots, 2t_0, (t_0 > \alpha)\ m = 0, \cdots, 2t_0 n - 1,\ j = 1, \cdots, n\rangle$,which shows that there do not exist subgroups $B_j^{(\alpha)\prime} (j = 1, \cdots, n)$ of $B^{(\alpha)\prime}$ such that $B^{(\alpha)\prime} = \oplus_{j=1}^{n} B_j^{(\alpha)\prime}$, $B_j^{(\alpha)\prime}/(B_j^{(\alpha)\prime} \cap A) \cong \mathbf{Q}'$, where $A = \oplus_{i \in \mathbf{N}} A_{p_i}$, $A_{p_i} = \oplus_{m=0}^{\infty}\langle a_{p_i m}\rangle$.

Also let $\widetilde{B}'_{p_l} = \langle \tilde{a}_{p_l m}^{[l]},\ \tilde{b}_{jp_i^t}^{[l]}$ for $i \in \mathbf{N}, t = 0, \cdots, 2t_0 (t_0 > \alpha),\ m = 0, \cdots, 2t_0 n - 1,\ j = 1, \cdots, n\rangle$ be a subgroup of $\widetilde{B}_{p_l}$ for any p_l, which yields a direct sum decomposition $\widetilde{B}'_{p_l} = \widetilde{C}'_{p_l} \oplus \widetilde{A}'_{p_l}$ with $\widetilde{C}'_{p_l} = \langle \tilde{b}_{jp_i^t}^{[l]\prime}$ for $i \in \mathbf{N}, t = 0, \cdots, 2t_0,\ j = 1, \cdots, n\rangle$ and $\widetilde{A}'_{p_l} = \oplus_{m=0}^{2t_0 n-1}\langle \tilde{a}_{p_l m}^{[l]}\rangle$, where $\tilde{b}_{jp_l^t}^{[l]\prime} = \tilde{b}_{jp_l^t}^{[l]} - \delta_{>}(2t_0, t)\sum_{s=t+1}^{2t_0}\sum_{m=(s-1)n}^{sn-1}\chi_j^{(0)}(m)p_l^{s-(t+1)}\tilde{a}_{p_l m}^{[l]}$, $\tilde{b}_{jp_i^t}^{[l]\prime} = \tilde{b}_{jp_i^t}^{[l]} - \sum_{s=1}^{2t_0}\sum_{m=(s-1)n}^{sn-1} r_{jp_i^t}^{(s)}(m)p_l^{s-1}\tilde{a}_{p_l m}^{[l]}$ $(l \neq i \in \mathbf{N})$.

Then, $\mathcal{C}$-representations of $B^{(\alpha)\prime}$ ($\alpha \in \mathbf{N_0}$) with respect to representative functions are $B^{(\alpha)\prime +} = B\big(\mathcal{C}',\ \left[\tilde{g}_{p_l}^{(\alpha)\prime}(u)\right]_{u \in C', l \in \mathbf{N}}\big)$ with $\mathcal{C}' = \big(C',\ \left[(\widetilde{B}'_{p_l}, \widetilde{\kappa}_{p_l})\right]_{l \in \mathbf{N}}\big)$ and $\tilde{g}_{p_l}^{(\alpha)\prime}(u) = \tilde{g}_{p_l}^{(0)\prime}(u) + \tilde{h}_{p_l}^{(\alpha)\prime}(u)$, where $\tilde{g}_{p_l}^{(0)\prime}(u) = \sum_{j=1}^{n}(s_j\tilde{b}_{j1}^{[l]\prime} + \sum_{i=1}^{I_j} r_{ji}\tilde{b}_{jp_i^{\alpha_{ji}}}^{[l]\prime})$,

$$\tilde{h}_{p_l}^{(\alpha)\prime}(u) = \sum_{j=1}^{n}\lfloor\sum_{s=1}^{2t_0}\sum_{m=(s-1)n}^{sn-1}\{s_j\chi_j^{(0)}(m) + \sum_{i=1}^{I_j}(1-\delta_{il})r_{ji}r_{jp_i^{\alpha_{ji}}}^{(s)}(m)\}p_l^{s-1}\tilde{a}_{p_l m}^{[l]}$$
$$+\sum_{i=1}^{I_j}\delta_{>}(2t_0, \alpha_{ji})\sum_{s=\alpha_{ji}+1}^{2t_0}\sum_{m=(s-1)n}^{sn-1}\delta_{il}r_{ji}\chi_j^{(0)}(m)p_l^{s-(\alpha_{ji}+1)}\tilde{a}_{p_l m}^{[l]}\rfloor - \tilde{h}_{p_l}^{(\alpha)}(u)$$

for $u = \sum_{j=1}^{n}\tau_j(s_j + \sum_{i=1}^{I_j}\frac{r_{ji}}{p_i^{\alpha_{ji}}}) \in C'$.

Further, $\widetilde{B}'_{p_l,p} = (\oplus_{j=1}^{n}\langle \tilde{g}_{p_l}^{(\alpha)\prime}(\tau_j(\frac{1}{p^{2t_0}}))\rangle) \oplus (\oplus_{m=0}^{2t_0 n-1}\langle \delta_{i_0 l}\tilde{a}_{p_l m}^{[l]}\rangle)$ is a p-basic subgroup of $\widetilde{B}'_{p_l}$, where

$$\tilde{g}_{p_l}^{(\alpha)\prime}(\tau_j(\tfrac{1}{p^{2t_0}})) = \tilde{b}_{jp^{2t_0}}^{[l]\prime} + \delta_{>}(\alpha, 0)\sum_{s=1}^{\alpha}\sum_{m=(s-1)n}^{sn-3}(1-\delta_{i_0 l})r_{jp^{2t_0}}^{(s)}(m)p_l^{s-1}\tilde{a}_{p_l m}^{[l]}$$
$$+\sum_{s=\alpha+1}^{2t_0}\sum_{m=(s-1)n}^{sn-1}(1-\delta_{i_0 l})r_{jp^{2t_0}}^{(s)}(m)p_l^{s-1}\tilde{a}_{p_l m}^{[l]}$$

for $p = p_{i_0}, t_0 > \alpha$ and any p_l.

Then, there exists a global p-basic subgroup $B_p^{(\alpha)\prime} = (\oplus_{j=1}^{n}\langle b_{jp^{2t_0}}^{(\alpha)}\rangle) \oplus (\oplus_{m=0}^{2t_0 n-1}\langle a_{pm}\rangle)$ of $B^{(\alpha)\prime}$ such that $\widetilde{B}'_{p_l,p} = (\rho_{p_l}^{(\alpha)})^{-1}((B_p^{(\alpha)\prime} + A_{p_l}^*)/A_{p_l}^*)$ by an isomorphism $\rho_{p_l}^{(\alpha)}$ from $\widetilde{B}_{p_l}$ onto $\bar{B}_{p_l}^{(\alpha)}$ for any p_l, where $\bar{B}_{p_l}^{(\alpha)} = B^{(\alpha)}/A_{p_l}^*$, $A_{p_l}^* = \oplus_{i \neq l, i \in \mathbf{N}} A_{p_i}$.

On the other hand, $\widetilde{B}''_{p_l,p} = (\oplus_{j=1}^{n}\langle \tilde{b}_{jp^{2t_0}}^{[l]\prime}\rangle) \oplus (\oplus_{m=0}^{2t_0 n-1}\langle \delta_{i_0 l}\tilde{a}_{p_l m}^{[l]}\rangle)$ is a p-basic subgroup of $\widetilde{B}'_{p_l}$ for $p = p_{i_0}, t_0 > \alpha$ and any p_l, too. However, since $\tilde{b}_{jp^{2t_0}}^{[l]\prime} - \tilde{g}_{p_l}^{(\alpha)\prime}(\tau_j(\frac{1}{p^{2t_0}})) \neq \tilde{0}^{[l]}$ with $\widetilde{\kappa}_{p_l}^{-1}(\tilde{b}_{jp^{2t_0}}^{[l]\prime} + \widetilde{A}_{p_l}) = \tau_j(\frac{1}{p^{2t_0}}) \in C'$ for $i_0 \neq l \in \mathbf{N}$, there does not exist a global p-basic subgroup $B_p^{(\alpha)\prime\prime}$ of $B^{(\alpha)\prime}$ such that $\widetilde{B}''_{p_l,p} = (\rho_{p_l}^{(\alpha)})^{-1}((B_p^{(\alpha)\prime\prime} + A_{p_l}^*)/A_{p_l}^*)$ for any p_l.

References

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