Local-Global Studies on Mixed Abelian Groups XIX

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Let $B$ be an arbitrary mixed group, and denote by $T(B)$ the torsion part of $B$. Then $A = T(B)$ can be decomposed as the internal direct sum $A = \oplus_{p \in \mathbf{P}} A_p$, where $A_p (p \in \mathbf{P})$ is the $p$-component of $A$. Also let $C$ be a torsion-free group such that $C \cong B/A$, and $\kappa$ an isomorphism from $C$ onto $B/A$. For a subgroup $B'$ of $B$ which is not contained in $A$, we put

$$C' = \kappa^{-1}((B' + A)/A), \quad A' = T(B').$$

In the cosets of $B'$ mod $A'$, take a complete set $[b\lambda]_{\lambda \in \Lambda}$ of representatives mod $A'$. Then in the cosets of $B' + A$ mod $A$, $[b\lambda]_{\lambda \in \Lambda}$ also is a complete set of representatives mod $A$.

If $\kappa^{-1}(b\lambda + A) = u \in C'$, then we define the function $g' : C' \rightarrow B'$ acting via $u \mapsto b\lambda$ and choose $g'(\mathbf{0}) = 0$ for $\mathbf{0} \in C$. This $g'$ is said to be a $C'$-function from $C'$ to $B'$ relative to $\kappa$. Especially, a $C$-function $g$ from $C$ to $B$ is identical with a representative function.

Also let $h_p$ be a function from $C$ to $A_p$ for any $p$. For any fixed element $u$ of $C$, if $h_p(u) = 0$ except for at most finitely many $p$'s, then we say that a $C$-common family $[h_p(u)]_{u \in C, p \in \mathbf{P}}$ satisfies "Finiteness Condition".

Next for any $p \in \mathbf{P}$, put $A_p' = \oplus_{q \neq p} A_q \oplus A_p, \quad B_p' = B/A_p', \quad B_p'' = (B' + A_p')/A_p'$, then we call $B_p''$ the $p$-localization of a global subgroup $B'$ of $B$. Also put $\overline{g_p}(u) = g(u) + A_p'$ ($\forall u \in C$), and let $\pi_{p\kappa}$ be the canonical isomorphism from $B/A$ onto $B/A_p'/A_p'' = B_p'/\overline{A_p'}$ ($\overline{A_p'} = A'/A_p'$).

Now let $C'$ be a fixed subgroup of $C$, and $B_p''$ a subgroup of $B_p''$ such that

$$C' = (\pi_{p\kappa})^{-1}((B_p'' + A_p)/A_p)$$

for any $p$.

From the studies before, we have already obtained the following theorem.

**Theorem** There exists a unique global subgroup $B'$ of $B$ so as to be the given $B_p'$ as its $p$-localization, i.e., $\overline{B_p'} = \overline{B_p''}$ for any $p$ if and only if we can take a $C'$-function $\overline{g_p}'$ from $C'$ to $\overline{B_p'}$ relative to $\pi_{p\kappa}$ for any $p$ such that $[\overline{g_p}'(u)]_{u \in C', p \in \mathbf{P}}$ satisfies the finiteness condition, where $\overline{g_p}'(u) = \overline{g_p}'(u) - \overline{g_p}(u)$ ($\forall u \in C'$).

Let $B_p$ be a $p$-basic subgroup of $B$. Then $(\overline{B_p})_q = (B_p + A_q)/A_q$ becomes a $p$-basic subgroup of $\overline{B_q} = B/A_q$ for any $q \in \mathbf{P}$. Especially $B_p$ is isomorphic to $(\overline{B_p})_p$.

Conversely, let $\overline{B_p', \overline{B_p}}' = (\oplus_{i \in I} \langle c_{pi} \rangle) \oplus (\oplus_{j \in J} \langle a_{pj} \rangle)$ be a $p$-basic subgroup of $\overline{B_p}$ as the above $\overline{B_p'}$, and $\overline{B_p', \overline{B_p}}' = (\oplus_{i \in I} \langle c_{pi} \rangle) \oplus (\oplus_{j \in J} \langle a_{pj} \rangle)$ be a $p$-basic subgroup of $\overline{B_q}$ as the above $\overline{B_q}$ such that $(\pi_{p\kappa})^{-1}(c_{pi}) = (\pi_{q\kappa})^{-1}(c_{qi}) = u_i \in C'$ and $a_{pj} \in \overline{A_p'}$. And put $\overline{h_p}'(u_i) = \overline{c_{pi}} - \overline{g_p}(u_i)$ ($\forall u_i \in C'$).

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Proposition If $\lfloor \tilde{h}_p'((u_i)) \rfloor_{u_i \in \mathfrak{C}', p \in \mathbb{P}}$ satisfies the finiteness condition, then there exists a unique $p$-basic subgroup $B_p$ of $B$ so as to be the given $B_{p,q}$ as its $p$-localization, i.e., $(B_p)_q = \tilde{B}_{p,q}$ for any $q \in \mathbb{P}$.

The following example indicates that for the above given $\lfloor \tilde{B}_p' \rfloor_{p \in \mathbb{P}}$, there does not always exist a subgroup $B'$ of $B$ satisfying $\overline{B}_p = \tilde{B}'_p$ ($\forall p \in \mathbb{P}$).

We intend to present a group $G$ such that

(1) the $p$-localization $G_p$ of $G$ splits for any $p$, but $G$ does not split.

(2) for a $p$-basic subgroup $\overline{B}_{p,q}$ of $G_q$ ($\forall q \in \mathbb{P}$), there is not a $p$-basic subgroup $B_p$ of $G$ satisfying $\overline{(B_p)}_q = B_{p,q}$ for any $q$.

Example We refer to [1, Vol. II, Example 2, p.186]. Define $T = \oplus_{p \in \mathbb{P}}(a_p)$ with $o(a_p) = p$, then $T$ is the torsion part of $\prod_{p \in \mathbb{P}}(a_p)$. Consider $b_0 = (a_2, a_3, a_5, \ldots, a_p, \ldots) \in \prod_{p \in \mathbb{P}}(a_p)$. For $p \neq q$, the equation $xq = a_p$ is uniquely solvable in $(a_p)$, thus $\prod_{p \in \mathbb{P}}(a_p)$ contains unique elements $b_p$ ($p \in \mathbb{P}$) such that

$p b_p = (a_2, a_3, \ldots, a_q, 0, a_{q+1}, \ldots) = b_0 - a_p$.

Put $G = \langle T, b_2, b_3, b_5, \ldots, b_p, \ldots \rangle$ in $\prod_{p \in \mathbb{P}}(a_p)$ and $C = G/T = \langle b_2 + T, b_3 + T, b_5 + T, \ldots, b_p + T, \ldots \rangle$, then $T$ is the torsion part of $G$ and $C$ is a torsion-free group. From $T_p = \oplus_{p \neq q}(a_q)$, it follows that $G_p = G/T_p = \langle a_p + T_p, b_2 + T_p, b_3 + T_p, \ldots, b_p + T_p, \ldots \rangle$. Since the torsion part $T_p = T/T_p = \langle a_p + T_p \rangle$ of $G_p$ is a bounded group, $G_p$ splits, i.e.,

$\overline{G}_p = \overline{G}_p \oplus \overline{T}_p$,

where $\overline{C}_p = \langle b_p + T_p, [b_q - a_p^{(q)} + T_p]_{p \neq q \in \mathbb{P}} \rangle$ with a relation $qa_p^{(q)} = a_p$ for certain $a_p^{(q)} \in (a_p)$.

However, $G$ does not split. Also $\overline{B}_{p,p} = \langle b_0 + T_p \rangle$ and $\overline{B}_{p,q} = \langle b_p - a_q^{(p)} + T_q \rangle$ is a $p$-basic subgroup of $\overline{G}_p$ and $\overline{B}_{p,q}$ is the torsion part of $\overline{G}_q$. Let $\kappa$ be the identity automorphism of $C = G/T$, and $\overline{\kappa}$ the canonical isomorphism from $G/T$ onto $G/T_p/T/T_p$ for any $p$. And for any coset $b = b + T$ ($b \in C$) of $G$ mod $T$, take a representative $g(b) = b + \sum_{p \in \mathbb{P}}d_p \langle d_p \in (a_p) \rangle$ of $b$, where $g(0) = 0$ for $0 \in C$. Further put $\overline{\sigma}_p(b) = g(b) + T_p$, then $\overline{\sigma}_p$ is a $C$-function from $C$ to $\overline{G}_p$ relative to $\overline{\sigma}_p \kappa$. Here $\overline{\sigma}_p(b_0)$ can be written in the form

$g(b_0) + T_p = (b_0 - a_p + T_p) + (a_p + d_p + T_p)$

with $\tilde{g}_p(b_0) = b_0 - a_p + T_p = p(b_2 + T) \in \overline{B}_{p,p} \subset \overline{C}_p$ and $\tilde{g}_p(b_0) = a_p + d_p + T_p \in \overline{T}_p$. Also $\tilde{g}_q(b_0) = b_0 - a_q + T_q$ and $\tilde{g}_q(b_0) = a_q + d_q + T_q \in \overline{T}_q$.

Therefore $\tilde{g}_p(b_0) - \overline{\sigma}_p(b_0) = -\tilde{g}_p(b_0)$ and $\tilde{g}_p(b_0) = a_p + T_p \neq 0$ for $p$ such that $d_p = 0$, which shows there are not subgroups $B_p$, $G'$ of $G$ such that

$\overline{B}_{p,q} = (B_p + T_q)/T_q$, $\overline{C}_q = (G' + T_q)/T_q$ ($\forall q \in \mathbb{P}$)

respectively, by Theorem.

References

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