P-8

Fixed point theorems for nonexpansive mappings in a vector lattice

Toshiharu Kawasaki [1] and Toshikazu Watanabe [2]

1 Topology in a vector lattice

First we introduce a topology in a vector lattice introduced by [2]; see also [4, 5].

Let X be a vector lattice. $e \in X$ is said to be an unit if $e \wedge x > 0$ for any $x \in X$ with $x > 0$. Let $\mathcal{K}_X$ be the class of units of X. In case where X is the set of real numbers $\mathbf{R}$, $\mathcal{K}_{\mathbf{R}}$ is the set of positive real numbers. Let X be a vector lattice with unit and let Y be a subset of X. Y is said to be open if for any $x \in Y$ and for any $e \in \mathcal{K}_X$ there exists $\varepsilon \in \mathcal{K}_{\mathbf{R}}$ such that $[x - \varepsilon e, x + \varepsilon e] \subset Y$. Let $\mathcal{O}_X$ be the class of open subsets of X.

A vector lattice is said to be Archimedean if it holds that $x = 0$ whenever there exists $y \in X$ with $y \geq 0$ such that $0 \leq rx \leq y$ for any $r \in \mathcal{K}_{\mathbf{R}}$.

Let X be an Archimedean vector lattice. Then there exists a positive homomorphism f from X into $\mathbf{R}$, that is, f satisfies the following conditions:

(H1) $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$ for any $x, y \in X$ and for any $\alpha, \beta \in \mathbf{R}$;

(H2) $f(x) \geq 0$ for any $x \in X$ with $x \geq 0$;

see [5]*Example 3.1. Suppose that there exists a homomorphism f from X into $\mathbf{R}$ satisfying the following condition instead of (H2):

$(\mathrm{H2})^s$ $f(x) > 0$ for any $x \in X$ with $x > 0$.

2 Fixed point theorem for a nonexpansive mapping

Let X be a vector lattice and Y a subset of X. A mapping f from Y into Y is said to be nonexpansive if $|f(x) - f(y)| \leq |x - y|$ for any $x, y \in Y$. In this section we consider a fixed point theorem for a nonexpansive mapping.

Let X be a Hausdorff Archimedean vector lattice with unit and Y a subset of X. We say that Y has the normal structure if for any compact convex subset K, which contains two points at least, of Y there exists $x \in K$ such that

$$\bigvee_{y \in K} |x - y| < \bigvee_{x, y \in K} |x - y|.$$

Theorem 2.1. *Let X be a Hausdorff Archimedean vector lattice with unit and K a non-empty compact convex subset of X. Suppose that K has the normal structure. Then every nonexpansive mapping from K into K has a fixed point.*

3 Fixed point theorem for the commutative family of nonexpansive mappings

For any nonexpansive mapping f from K into K let $F_K(f)$ be the set of fixed points of f.

Theorem 3.1. *Let X be a Hausdorff Archimedean vector lattice with unit, K a compact convex subset of X and $\{f_i \mid i = 1, \cdots, n\}$ the finite commutative family of nonexpansive mappings from K into K. Suppose that there exists a homomorphism from X into $\mathbf{R}$ satisfying condition $(\mathrm{H2})^s$ and K has the normal structure. Then $\bigcap_{i=1}^{n} F_K(f_i)$ is non-empty.*

Theorem 3.2. *Let X be a Hausdorff Archimedean vector lattice with unit, K a compact convex subset of X and $\{f_i \mid i \in I\}$ the commutative family of nonexpansive mappings from K into K. Suppose that there exists a homomorphism from X into $\mathbf{R}$ satisfying condition $(\mathrm{H2})^s$ and K has the normal structure. Then $\bigcap_{i \in I} F_K(f_i)$ is non-empty.*

1 日立製作所　2 日大理工・非常勤

References

[1] R. Cristescu, Topological Vector Spaces, Noordhoff International Publishing, Leyden, 1977. Zbl 0345.46001

[2] T. Kawasaki, *Denjoy integral and Henstock-Kurzweil integral in vector lattices, I, II*, Czechoslovak Math. J., **59** (2009), no. 2, 381–399, 401–417.

[3] T. Kawasaki, M. Toyoda, Masashi and T. Watanabe, *Fixed point theorem for set-valued mapping in a Riesz space*, Memoirs of the Faculty of Engineering, Tamagawa University, **44** (2009), 81–85 (in Japanese).

[4] T. Kawasaki, M. Toyoda and T. Watanabe, *Takahashi's and Fan-Browder's fixed point theorems in a vector lattice*, Journal of Nonlinear and Convex Analysis, **10** (2009), no. 3, 455–461.

[5] T. Kawasaki, M. Toyoda and T. Watanabe, *Schauder-Tychonoff's fixed point theorems in a vector lattice*, Fixed Point Theory, **11** (2009), no. 1, 37–44.

[6] W. A. J. Luxemburg and A. C. Zaanen, Riesz Spaces, North-Holland, Amsterdam, 1971. Zbl 0231.46014

[7] W. Takahashi, *Fixed point, minimax, and Hahn-Banach theorems*, Proc. of Symposia in Pure Math., **45** (1986), no. 2, 419–427. Zbl 0636.47048

[8] W. Takahashi, Nonlinear Functional Analysis. Fixed Points Theory and its Applications, Yokohama Publishers, Yokohama, 2000. Zbl 0997.47002

[9] B. Z. Vulikh, Introduction to the Theory of Partially Orderd Spaces, Wolters-Noordhoff, Groningen, 1967. Zbl 0186.44601