P−9

Local-Global Studies on Mixed Abelian Groups IIX

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We intend to give two examples about $\mathcal{C}$-representations of mixed groups, which make their structures clear and drive the local-global properties on p-basic subgroups of those.

First, let $C = \langle \frac{1}{p} : p \in \mathbf{P} \rangle$ be a rational group and $B_p = \langle d_0^{(p)}, \left[d_q^{(p)}\right]_{q\in\mathbf{P}}, a_p^{(p)} \rangle$ a p-mixed group defined as follows : it is generated by elements $d_0^{(p)}, d_q^{(p)}$ $(\forall q \in \mathbf{P})$, $a_p^{(p)}$ such that $qd_q^{(p)} = d_0^{(p)}, pa_p^{(p)} = 0$. Then $B_p = C_p \oplus A_p$ with $C_p = \langle d_0^{(p)}, \left[d_q^{(p)}\right]_{q\in\mathbf{P}} \rangle$ and $A_p = \langle a_p^{(p)} \rangle$. And (B_p, κ_p) is a (C, κ_p)-p-mixed group, where κ_p is an isomorphism from C onto B_p/A_p acting via $\frac{m}{n} = s + \sum_{1\le i\le I} \frac{r_i}{p_i} \longmapsto s(d_0^{(p)} + A_p) + \sum_{1\le i\le I} r_i(d_{p_i}^{(p)} + A_p)$ ($n = \prod_{1\le i\le I} p_i, s \in \mathbf{Z}, 0 \le r_i < p_i$).

Next, to construct a mixed group B_1 such that the p-localization $\overline{B_{1p}}$ of B_1 is isomorphic to B_p for any p, we refer to [1, Vol. II, Example 2, p.186]. Define $A_1 = \oplus_{p\in\mathbf{P}} \langle a_p \rangle$ with $o(a_p) = p$, then A_1 is the torsion part of $\prod_{p\in\mathbf{P}} \langle a_p \rangle$. Consider $b_0 = (a_2, a_3, a_5, \ldots, a_p, \cdots) \in \prod_{p\in\mathbf{P}} \langle a_p \rangle$. For $p \ne q$, the equation $qx = a_p$ is uniquely solvable in $\langle a_p \rangle$, thus $\prod_{p\in\mathbf{P}} \langle a_p \rangle$ contains unique elements b_p ($p \in \mathbf{P}$) such that $pb_p = (a_2, a_3, \ldots, a_q, 0, a_{q'}, \cdots) = b_0 - a_p$. Put $B_1 = \langle A_1, b_2, b_3, b_5, \ldots, b_p, \cdots \rangle$ in $\prod_{p\in\mathbf{P}} \langle a_p \rangle$ and $\overline{b_p} = b_p + A_1$ for any p, then A_1 is the torsion part of B_1 and $B_1/A_1 = \langle \overline{b_2}, \overline{b_3}, \overline{b_5}, \ldots, \overline{b_p}, \cdots \rangle$ is a torsion-free group. For every $\frac{1}{p} \in C$, the correspondence $\frac{1}{p} \mapsto \overline{b_p}$ induces an isomorphism κ_1 from C onto B_1/A_1. From $A_{1p}^* = \oplus_{p\ne q\in\mathbf{P}} \langle a_q \rangle$, it follows that $\overline{B_{1p}} = B_1/A_{1p}^* = \langle a_p + A_{1p}^*, b_2 + A_{1p}^*, b_3 + A_{1p}^*, \ldots, b_q + A_{1p}^*, \cdots \rangle$. Since the torsion part $\overline{A_{1p}} = A_1/A_{1p}^* = \langle a_p + A_{1p}^* \rangle$ of $\overline{B_{1p}}$ is a bounded group, $\overline{B_{1p}}$ splits, i.e., $\overline{B_{1p}} = \widetilde{C_{1p}} \oplus \overline{A_{1p}}$, where $\widetilde{C_{1p}} = \langle b_0 - a_p + A_{1p}^*, \left[b_q - a_{p(q)} + A_{1p}^*\right]_{q\in\mathbf{P}} \rangle$ with relations $a_{p(p)} = 0$, $qa_{p(q)} = a_p$ for certain $a_{p(q)} \in \langle a_p \rangle$. However B_1 does not split. Then $(\overline{B_{1p}}, \overline{\kappa_{1p}}\kappa_1)$ is a $(C, \overline{\kappa_{1p}}\kappa_1)$-$p$-mixed group, where $\overline{\kappa_{1p}}$ is the canonical isomorphism from B_1/A_1 onto $B_1/A_{1p}^* \big/ A_1/A_{1p}^*$. And with any elements $b_0 - a_p + A_{1p}^*, b_q - a_{p(q)} + A_{1p}^*(q \in \mathbf{P}), a_p + A_{1p}^*$ of $\overline{B_{1p}}$, we associate elements $d_0^{(p)}, d_q^{(p)}(q \in \mathbf{P}), a_p^{(p)}$ of B_p respectively. This association gives rise to a C-isomorphism ρ_{1p} from $(\overline{B_{1p}}, \overline{\kappa_{1p}}\kappa_1)$ onto (B_p, κ_p) for any p.

Similarly, we construct another mixed group B_2 such that the p-localization $\overline{B_{2p}}$ of B_2 is isomorphic to B_p for any p. Let $B_2 = B_1/A_1 \oplus A_1$ be the external direct sum, and $\mu_2 : A_1 \to B_2$ an injection acting via $a \mapsto (\overline{0}, a)$ $(a \in A_1)$. Then the torsion part A_2 of B_2 can be decomposed as the internal direct sum $A_2 = \oplus_{p\in\mathbf{P}} A_{2p}$ $(A_{2p} = \mu_2(\langle a_p \rangle))$. For every $\frac{1}{p} \in C$, the correspondence $\frac{1}{p} \mapsto (\overline{b_p}, 0) + A_2$ induces an isomorphism κ_2 from C onto B_2/A_2. Also , for any p, put $A_{2p}^* = \oplus_{p\ne q\in\mathbf{P}} A_{2q}, \overline{B_{2p}} = B_2/A_{2p}^*$, and let $\overline{\kappa_{2p}}$ be the canonical isomorphism from B_2/A_2 onto $B_2/A_{2p}^* \big/ A_2/A_{2p}^*$. Then $(\overline{B_{2p}}, \overline{\kappa_{2p}}\kappa_2)$ is a $(C, \overline{\kappa_{2p}}\kappa_2)$-$p$-mixed group. And with any elements $(\overline{b_0}, 0) + A_{2p}^*, (\overline{b_q}, 0) + A_{2p}^*, (q \in \mathbf{P}), \mu_2 a_p + A_{2p}^*$ of $\overline{B_{2p}}$, we associate elements $d_0^{(p)}, d_q^{(p)}(q \in \mathbf{P}), a_p^{(p)}$ of B_p respectively. This association gives rise to a C-isomorphism ρ_{2p} from $(\overline{B_{2p}}, \overline{\kappa_{2p}}\kappa_2)$ onto (B_p, κ_p) for any p. Therefore, $(\overline{B_{1p}}, \overline{\kappa_{1p}}\kappa_1)$ and $(\overline{B_{2p}}, \overline{\kappa_{2p}}\kappa_2)$ are C-isomorphic.

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Thereafter we consider $\mathcal{C}$-representations of B_1 and B_2.

Take a representative $g_1(\frac{m}{n}) = sb_0 + \sum_{1\le i\le I} r_i b_{p_i}$ of the coset $\kappa_1(\frac{m}{n})$, if $\frac{m}{n} = s + \sum_{1\le i\le I}\frac{r_i}{p_i}$ for $\frac{m}{n}\in C$. Further put $\overline{g_{1p}}(\frac{m}{n}) = g_{1p}(\frac{m}{n}) + A^*_{1p}$, then $\overline{g_{1p}}$ is a C-function from C to $\overline{B_{1p}}$ relative to $\overline{\kappa_{1p}}\kappa_1$ and the following holds $\rho_{1p}\overline{g_{1p}}\left(\frac{m}{n}\right) = sd_0^{(p)} + \sum_{1\le i\le I} r_i d_{p_i}^{(p)} + sa_p^{(p)} + \sum_{1\le i\ne k\le I} r_i a_{p_i}^{(p)}$ ($p = p_k, ; p_i a_{p_i}^{(p)} = a_p^{(p)}, a_{p_i}^{(p)} \in \langle a_p^{(p)}\rangle$). The $g_{1p} = \rho_{1p}\overline{g_{1p}}$ becomes a C-function from C to B_p relative to κ_p, which yields the factor set f_{1p} on C to A_p as follows : $f_{1p}\left(\frac{m'}{n'}, \frac{m''}{n''}\right) = f_{1p}\left(s' + \sum_{1\le i\le I}\frac{r'_i}{p_i},\ s'' + \sum_{1\le i\le I}\frac{r''_i}{p_i}\right) = -\delta_k a_{p_k}^{(p_k)}\ \left(0\le r'_i, r''_i < p_i;\ p = p_k,\ \delta_k = \left[\frac{r'_k + r''_k}{p_k}\right]\right)$.

Similarly, take a representative $g_2(\frac{m}{n}) = s(\overline{b_0}, 0) + \sum_{1\le i\le I} r_i(\overline{b_{p_i}}, 0)$ of the coset $\kappa_2(\frac{m}{n})$, if $\frac{m}{n} = s + \sum_{1\le i\le I}\frac{r_i}{p_i}$ for $\frac{m}{n}\in C$. Further put $\overline{g_{2p}}(\frac{m}{n}) = g_{2p}(\frac{m}{n}) + A^*_{2p}$, then $\overline{g_{2p}}$ is a C-function from C to $\overline{B_{2p}}$ relative to $\overline{\kappa_{2p}}\kappa_2$ and the following holds $\rho_{2p}\overline{g_{2p}}\left(\frac{m}{n}\right) = sd_0^{(p)} + \sum_{1\le i\le I} r_i d_{p_i}^{(p)}$. The $g_{2p} = \rho_{2p}\overline{g_{2p}}$ becomes a C-function from C to B_p relative to κ_p, which yields the factor set f_{2p} on C to A_p as follows : $f_{2p}\left(\frac{m'}{n'}, \frac{m''}{n''}\right) = 0^{(p)}$.

Next put $\mathcal{C} = \big(C,\ [(B_p, \kappa_p)]_{p\in\mathbf{P}}\big)$, $B = \prod_{p\in\mathbf{P}} B_p$, $A = \oplus_{p\in\mathbf{P}} A_p$, and let μ_p be a coordinate injection from A_p into A acting via $a_p \mapsto (0,\cdots,0,a_p,0,\cdots)$. In B, take $g_1^+(\frac{m}{n}) = \big(g_{1p}(\frac{m}{n})\big)_{p\in\mathbf{P}}$, $g_2^+(\frac{m}{n}) = \big(g_{2p}(\frac{m}{n})\big)_{p\in\mathbf{P}}$, then we can construct the mixed groups B_1^+, B_2^+ generated by adjoining $\big[g_1^+(\frac{m}{n})\big]_{\frac{m}{n}\in C}$, $\big[g_2^+(\frac{m}{n})\big]_{\frac{m}{n}\in C}$ to A respectively, and shall write $B_1^+ = \left\langle A, \big[g_1^+(\frac{m}{n})\big]_{\frac{m}{n}\in C}\right\rangle$, $B_2^+ = \left\langle A, \big[g_2^+(\frac{m}{n})\big]_{\frac{m}{n}\in C}\right\rangle$. Here B_1^+, B_2^+ are isomorphic to B_1, B_2 respectively, and $B_1^+ = B\big(\mathcal{C},\ \big[g_{1p}(\frac{m}{n})\big]_{\frac{m}{n}\in C, p\in\mathbf{P}}\big)$, $B_2^+ = B\big(\mathcal{C},\ \big[g_{2p}(\frac{m}{n})\big]_{\frac{m}{n}\in C, p\in\mathbf{P}}\big)$ are $\mathcal{C}$-representations of B_1, B_2 with respect to C-functions, respectively.

On the other hand, we construct a group B_1^- as the set of all pairs $\big(\frac{m}{n}, a\big)_1 \in C\times A$ with the operation $\left(\frac{m'}{n'}, a'\right)_1 + \left(\frac{m''}{n''}, a''\right)_1 = \left(\frac{m'}{n'} + \frac{m''}{n''}, a' + a'' + f_1(\frac{m'}{n'}, \frac{m''}{n''})\right)_1$, where f_1 is a factor set on C to A as follows : $f_1\left(\frac{m'}{n'}, \frac{m''}{n''}\right) = \sum_{p\in\mathbf{P}}\mu_p f_{1p}\left(s' + \sum_{1\le i\le I}\frac{r'_i}{p_i},\ s'' + \sum_{1\le i\le I}\frac{r''_i}{p_i}\right) = \sum_{1\le k\le I}\mu_{p_k}\left(-\delta_k a_{p_k}^{(p_k)}\right)$. Similarly, we construct a group B_2^- as the set of all pairs $\big(\frac{m}{n}, a\big)_2 \in C\times A$ with the operation $\left(\frac{m'}{n'}, a'\right)_2 + \left(\frac{m''}{n''}, a''\right)_2 = \left(\frac{m'}{n'} + \frac{m''}{n''}, a' + a'' + f_2(\frac{m'}{n'}, \frac{m''}{n''})\right)_2$, where f_2 is a factor set on C to A as follows : $f_2\left(\frac{m'}{n'}, \frac{m''}{n''}\right) = \sum_{p\in\mathbf{P}}\mu_p f_{2p}\left(\frac{m'}{n'}, \frac{m''}{n''}\right) = \sum_{p\in\mathbf{P}}\mu_p 0^{(p)} = 0$. Here B_1^-, B_2^- are isomorphic to B_1, B_2 respectively, and $B_1^- = B\big(\mathcal{C},\ \left[f_{1p}\left(\frac{m'}{n'}, \frac{m''}{n''}\right)\right]_{\frac{m'}{n'}, \frac{m''}{n''}\in C, p\in\mathbf{P}}\big)$, $B_2^- = B\big(\mathcal{C},\ \left[f_{2p}\left(\frac{m'}{n'}, \frac{m''}{n''}\right)\right]_{\frac{m'}{n'}, \frac{m''}{n''}\in C, p\in\mathbf{P}}\big)$ are $\mathcal{C}$-representations of B_1, B_2 with respect to factor sets, respectively.

Finally, we wish to apply this method to the study of p-basic subgroups.

Now, $B_{pp} = \left\langle d_p^{(p)}\right\rangle \oplus \left\langle a_p^{(p)}\right\rangle$ is a p-basic subgroup of B_p, and $B_{pq} = \left\langle d_p^{(q)}\right\rangle$ is a p-basic subgroup of B_q for $p\ne q\in\mathbf{P}$. With $\kappa_p^{-1}(d_p^{(q)} + A_q) = \frac{1}{p}\in C$ and a C-function g_{1q}, the following holds $h_{1q}(\frac{1}{p}) = d_p^{(q)} - g_{1q}(\frac{1}{p}) = d_p^{(q)} - (d_p^{(q)} + a_p^{(q)}) = -a_p^{(q)}$ for any $q(\ne p)$ $(pa_p^{(q)} = a_q^{(q)}, a_p^{(q)} \in \left\langle a_q^{(q)}\right\rangle)$, which shows that there does not exist a global p-basic subgroup B_{1p} of B_1 such that $\overline{(B_{1p})_p} = (B_{1p} + A^*_{1p})/A^*_{1p} = \rho_{1p}^{-1}(B_{pp}) = \langle b_p + A^*_{1p}\rangle \oplus \langle a_p + A^*_{1p}\rangle$, $\overline{(B_{1p})_q} = (B_{1p} + A^*_{1q})/A^*_{1q} = \rho_{1q}^{-1}(B_{pq}) = \langle b_p - a_{q(p)} + A^*_{1q}\rangle$. However, since $h_{2q}(\frac{1}{p}) = d_p^{(q)} - g_{2q}(\frac{1}{p}) = d_p^{(q)} - d_p^{(q)} = 0$ for any q, there exists a global p-basic subgroup B_{2p} of B_2 such that $\overline{(B_{2p})_p} = \rho_{2p}^{-1}(B_{pp}) = \langle(\overline{b_p}, 0) + A^*_{2p}\rangle \oplus \langle \mu_2 a_p + A^*_{2p}\rangle$, $\overline{(B_{2p})_q} = \rho_{2q}^{-1}(B_{pq}) = \langle(\overline{b_p}, 0) + A^*_{2q}\rangle$.

References

[1] L.Fuchs Infinite abelian groups, Vols. I, II, Academic Press, New York-London (1970, 1973).