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$\mathcal{C}$-representations of Mixed Abelian Groups

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We intend to give a mixed group B_1 which is an extension of a direct sum $\oplus_{p\in\mathbf{P}}\langle a_p\rangle$ by a completely decomposable group C of finite rank, where $\langle a_p\rangle$ is a cyclic group of order p^t. Further we obtain $\mathcal{C}$-representations of B_1 to investigate the structures of B_1 and derive the local-global properties on p-basic subgroups of B_1.

First, let $p_1 = 2, p_2 = 3, \ldots$ be a listing of the prime numbers in increasing order and $(l_{j1}, l_{j2}, \ldots, l_{ji}, \ldots)$ a sequence of nonnegative integers for $j = 1, \ldots, n$. Define C by the direct sum of rational groups $C_j = \langle \frac{1}{p_i}, \ldots, \frac{1}{p_i^{l_{ji}}}\rangle_{i\in\mathbf{N}}$ of type $\mathbf{t}(C_j) = (l_{j1}, l_{j2}, \ldots, l_{ji}, \ldots)$, one for each j, i.e., $C = \oplus_{j=1,\ldots,n} C_j$. Then any element $\frac{m_j}{n_j}$ of C_j can be written in the form $\frac{m_j}{n_j} = s_j + \sum_{1\le i\le I_j} \frac{r_{ji}}{p_i^{\alpha_i}}$, where $n_j = \prod_{1\le i\le I_j} p_i^{\alpha_i} (0 \le \alpha_i \le l_{ji}), m_j, s_j, r_{ji} \in \mathbf{Z}, 0 \le r_{ji} < p_i^{\alpha_i}$. Also let $B^{(p_l)} = \langle d^{(l)}_{j0}, d^{(l)}_{jp_i}, \ldots, d^{(l)}_{j{p_i}^{l_{ji}}}, a^{(l)}_{p_l}\rangle_{i\in\mathbf{N}, j=1,\ldots,n}$ be a p_l-mixed group defined as follows : it is generated by elements $d^{(l)}_{j0}, d^{(l)}_{j{p_i}^k}$ $(1 \le k \le l_{ji}, i \in \mathbf{N})$, $a^{(l)}_{p_l}$ such that $p_i d^{(l)}_{j{p_i}^k} = d^{(l)}_{j{p_i}^{k-1}} (2 \le k \le l_{ji}), p_i d^{(l)}_{jp_i} = d^{(l)}_{j0}, p_l^{t_l} a^{(l)}_{p_l} = 0^{(l)}$. Then $B^{(p_l)} = C^{(p_l)} \oplus A^{(p_l)}$ with $C^{(p_l)} = \langle d^{(l)}_{j0}, d^{(l)}_{jp_i}, \ldots, d^{(l)}_{j{p_i}^{l_{ji}}}\rangle_{i\in\mathbf{N}, j=1,\ldots,n}$ and $A^{(p_l)} = \langle a^{(l)}_{p_l}\rangle$. The correspondence
$\sum_{j=1,\ldots,n} \nu_j(s_j + \sum_{1\le i\le I_j} \frac{r_{ji}}{p_i^{\alpha_i}}) \longmapsto \sum_{j=1,\ldots,n} \left\{ s_j(d^{(l)}_{j0} + A^{(p_l)}) + \sum_{1\le i\le I_j} r_{ji}(d^{(l)}_{jp_i^{\alpha_i}} + A^{(p_l)})\right\}$
induces an isomorphism $\kappa_{(p_l)}$ from C onto $B^{(p_l)}/A^{(p_l)}$, where the coordinate injection $\nu_j : \frac{m_j}{n_j} \longmapsto (0, \ldots, 0, \frac{m_j}{n_j}, 0, \ldots, 0) \in C$. And $(B^{(p_l)}, \kappa_{(p_l)})$ is a $(C, \kappa_{(p_l)})$-p_l-mixed group.

Next, we shall construct a mixed group B_1 such that the p_l-localization $\overline{B_{1p_l}}$ of B_1 is isomorphic to $B^{(p_l)}$ for any p_l. Define $A_1 = \oplus_{l\in\mathbf{N}}\langle a_{p_l}\rangle$ with $o(a_{p_l}) = p_l^{t_l}$, then A_1 is the torsion part of $\prod_{l\in\mathbf{N}}\langle a_{p_l}\rangle$. Consider $b_{j0} = (\widetilde{\epsilon_{jl}} a_{p_l})_{l\in\mathbf{N}} \in \prod_{l\in\mathbf{N}}\langle a_{p_l}\rangle$, where $\widetilde{\epsilon_{jl}} = 0$ if $l \equiv j \bmod n$ or $\widetilde{\epsilon_{jl}} = 1$ if $l \not\equiv j \bmod n$. For $i \ne l$, the equation $p_i^k x = \widetilde{\epsilon_{jl}} a_{p_l}$ is uniquely solvable in $\langle \widetilde{\epsilon_{jl}} a_{p_l}\rangle$, thus $\prod_{l\in\mathbf{N}}\langle a_{p_l}\rangle$ contains unique elements $b_{jp_i^k}$ $(i \in \mathbf{N}, j = 1, \ldots, n)$ such that $p_i b_{jp_i^k} = b_{jp_i^{k-1}}, p_i b_{jp_i} = b_{j0} - \widetilde{\epsilon_{ji}} a_{p_i}$. Put $B_1 = \langle A_1, b_{j0}, b_{jp_i}, \ldots, b_{j{p_i}^{l_{ji}}}\rangle_{i\in\mathbf{N}, j=1,\ldots,n}$ in $\prod_{l\in\mathbf{N}}\langle a_{p_l}\rangle$ and $\bar{b} = b + A_1$ for any $b \in B_1$, then A_1 is the torsion part of B_1 and $B_1/A_1 = \langle \overline{b_{j0}}, \overline{b_{jp_i}}, \ldots, \overline{b_{j{p_i}^{l_{ji}}}}\rangle_{i\in\mathbf{N}, j=1,\ldots,n}$ is a torsion-free group. For every $\nu_j(1), \nu_j(\frac{1}{p_i^k}) \in C$,the correspondence $\nu_j(1) \mapsto \overline{b_{j0}}, \nu_j(\frac{1}{p_i^k}) \mapsto \overline{b_{j{p_i}^k}}$ induces an isomorphism κ_1 from C onto B_1/A_1. From $A^*_{1p_l} = \oplus_{l\ne i\in\mathbf{N}}\langle a_{p_i}\rangle$, it follows that $\overline{B_{1p_l}} = B_1/A^*_{1p_l} = \langle a_{p_l} + A^*_{1p_l}, b_{j0} + A^*_{1p_l}, b_{jp_i} + A^*_{1p_l}, \ldots, b_{j{p_i}^{l_{ji}}} + A^*_{1p_l}\rangle_{i\in\mathbf{N}, j=1,\ldots,n}$. Since the torsion part $\overline{A_{1p_l}} = A_1/A^*_{1p_l} = \langle a_{p_l} + A^*_{1p_l}\rangle$ of $\overline{B_{1p_l}}$ is a bounded group, $\overline{B_{1p_l}}$ splits, i.e., $\overline{B_{1p_l}} = \widetilde{C_{1p_l}} \oplus \overline{A_{1p_l}}$, where $\widetilde{C_{1p_l}} = \oplus_{j=1,\ldots,n}\left\{\sum_{i\in\mathbf{N}}\langle b_{j0} - \widetilde{\epsilon_{jl}} a_{p_l} + A^*_{1p_l}, b_{jp_i} - \widetilde{r^{(l)}_{jp_i}}\widetilde{\epsilon_{jl}} a_{p_l} + A^*_{1p_l}, \ldots, b_{j{p_i}^{l_{ji}}} - \widetilde{r^{(l)}_{j{p_i}^{l_{ji}}}}\widetilde{\epsilon_{jl}} a_{p_l} + A^*_{1p_l}\rangle\right\}$ with relations $p_i^k \widetilde{r^{(l)}_{j{p_i}^k}} \equiv 1 \bmod p_l^{t_l}$ if $i \ne l$ or $\widetilde{r^{(l)}_{j{p_l}^k}} = 0$ for certain $\widetilde{r^{(l)}_{j{p_i}^k}} \in \mathbf{Z}$. However B_1 does not split. Then $(\overline{B_{1p_l}}, \overline{\kappa_{1p_l}}\kappa_1)$ is a $(C, \overline{\kappa_{1p_l}}\kappa_1)$-$p_l$-mixed group, where $\overline{\kappa_{1p_l}}$ is the canonical isomorphism from B_1/A_1 onto $B_1/A^*_{1p_l}\Big/ A_1/A^*_{1p_l}$.

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And with any elements $b_{j0} - \widetilde{\epsilon_{jl}} a_{p_l} + A^*_{1p_l}, b_{jp_i^k} - \widetilde{r^{(l)}_{jp_i^k}} \widetilde{\epsilon_{jl}} a_{p_l} + A^*_{1p_l} (j = 1, \dots, n,\ i \in \mathbf{N},\ k = 1, \dots, l_{ji}), a_{p_l} + A^*_{1p_l}$ of $\overline{B_{1p_l}}$, we associate elements $d^{(l)}_{j0}, d^{(l)}_{jp_i^k} (j = 1, \dots, n,\ i \in \mathbf{N},\ k = 1, \dots, l_{ji}), a^{(l)}_{p_l}$ of $B^{(p_l)}$ respectively. This association gives rise to a C-isomorphism ρ_{1p_l} from $(\overline{B_{1p_l}}, \overline{\kappa_{1p_l}} \kappa_1)$ onto $(B^{(p_l)}, \kappa_{(p_l)})$ for any p_l. Hence $\overline{B_{1p_l}}/\overline{A_{1p_l}}$ is the direct sum of rank-one groups, one of type $\mathbf{t}(C_j)$ for $j = 1, \dots, n$.

Thereafter, we consider $\mathcal{C}$-representations of B_1. For $u = \sum_{j=1,\dots,n} \nu_j (s_j + \sum_{1 \le i \le I_j} \frac{r_{ji}}{p_i^{\alpha_i}}) \in C$, choose a representative $g_1(u) = \sum_{j=1,\dots,n} \left\{ s_j b_{j0} + \sum_{1 \le i \le I_j} r_{ji} b_{jp_i^{\alpha_i}} \right\}$ of the coset $\kappa_1(u)$. Further put $\overline{g_{1p_l}}(u) = g_1(u) + A^*_{1p_l}$, then $\overline{g_{1p_l}}$ is a C-function from C to $\overline{B_{1p_l}}$ relative to $\overline{\kappa_{1p_l}} \kappa_1$. And the following holds $\rho_{1p_l} \overline{g_{1p_l}}(u) = \sum_{j=1,\dots,n} \left\{ s_j d^{(l)}_{j0} + \sum_{1 \le i \le I_j} r_{ji} d^{(l)}_{jp_i^{\alpha_i}} + s_j \widetilde{\epsilon_{jl}} a^{(l)}_{p_l} + \sum_{1 \le i \le I_j} r_{ji} \widetilde{r^{(l)}_{jp_i^{\alpha_i}}} \widetilde{\epsilon_{jl}} a^{(l)}_{p_l} \right\}$.
The $g^{(p_l)}_{(1)} = \rho_{1p_l} \overline{g_{1p_l}}$ becomes a C-function from C to $B^{(p_l)}$ relative to $\kappa_{(p_l)}$, which yields the factor set $f^{(p_l)}_{(1)}$ on C to $A^{(p_l)}$ as follows : $f^{(p_l)}_{(1)}(u', u'') = - \sum_{j=1,\dots,n} \left[\frac{r'_{jl} + r''_{jl}}{p_l^{\alpha_l}} \right] \widetilde{\epsilon_{jl}} a^{(l)}_{p_l}$ with Gauss' symbol for $u' = \sum_{j=1,\dots,n} \nu_j (s'_j + \sum_{1 \le i \le I_j} \frac{r'_{ji}}{p_i^{\alpha_i}})$ and $u'' = \sum_{j=1,\dots,n} \nu_j (s''_j + \sum_{1 \le i \le I_j} \frac{r''_{ji}}{p_i^{\alpha_i}})$ of C.

Next, put $\mathcal{C} = \big(C,\ [(B^{(p_l)}, \kappa_{(p_l)})]_{l \in \mathbf{N}}\big)$, $B = \prod_{l \in \mathbf{N}} B^{(p_l)}$, $A = \oplus_{l \in \mathbf{N}} A^{(p_l)}$, and let μ_{p_l} be a coordinate injection from $A^{(p_l)}$ into A acting via $a_{p_l} \mapsto (0, \cdots, 0, a_{p_l}, 0, \cdots)$.
In B, put $g^+_1(u) = \left(g^{(p_l)}_{(1)}(u) \right)_{l \in \mathbf{N}} \in \prod_{l \in \mathbf{N}} B^{(p_l)}$, then we construct a mixed group B^+_1 generated by adjoining $\left[g^+_1(u) \right]_{u \in C}$ to A, and write $B^+_1 = \left\langle A, \left[g^+_1(u) \right]_{u \in C} \right\rangle$. Here B^+_1 is isomorphic to B_1, and $B^+_1 = B\big(\mathcal{C},\ \left[g^{(p_l)}_{(1)}(u) \right]_{u \in C, l \in \mathbf{N}}\big)$ is a $\mathcal{C}$-representation of B_1 with respect to C-functions.

On the other hand, we construct a group B^-_1 as the set of all pairs $(u, a)_1 \in C \times A$ with the opration $(u', a')_1 + (u'', a'')_1 = \big(u' + u'', a' + a'' + f_{(1)}(u', u'')\big)_1$, where $f_{(1)}$ is a factor set on C to A as follows : $f_{(1)}(u', u'') = \sum_{l \in \mathbf{N}} \mu_{p_l} f^{(p_l)}_{(1)}(u', u'') = \sum_{l \in \mathbf{N}} \mu_{p_l} \left(- \sum_{j=1,\dots,n} \left[\frac{r'_{jl} + r''_{jl}}{p_l^{\alpha_l}} \right] \widetilde{\epsilon_{jl}} a^{(l)}_{p_l} \right)$.
Here B^-_1 is isomorphic to B_1, and $B^-_1 = B\big(\mathcal{C},\ \left[f^{(p_l)}_{(1)}(u', u'') \right]_{u', u'' \in C, l \in \mathbf{N}}\big)$ is a $\mathcal{C}$-representation of B_1 with respect to factor sets.

Finally, we wish to apply this method to the study of p-basic subgroups.

Now, $B^{(p_l)}_p = \left(\oplus_{j=1,\dots,n} \langle d^{(l)}_{jp^{l_j}} \rangle \right) \oplus \langle \delta_{li_0} a^{(l)}_{p_l} \rangle$ is a p-basic subgroup of $B^{(p_l)}$, where $p = p_{i_0}, l_j = l_{ji_0}$ and Kronecker's delta δ_{li_0}. With $\kappa^{-1}_{(p_l)}(d^{(l)}_{jp^{l_j}} + A^{(p_l)}) = \nu_j(\frac{1}{p^{l_j}}) \in C$ and C-function $g^{(p_l)}_{(1)}$, the following holds $h_{1p_l} \left(\nu_j(\frac{1}{p^{l_j}}) \right) = d^{(l)}_{jp^{l_j}} - g^{(p_l)}_{(1)} \left(\nu_j(\frac{1}{p^{l_j}}) \right) = d^{(l)}_{jp^{l_j}} - (d^{(l)}_{jp^{l_j}} + \widetilde{r^{(l)}_{jp^{l_j}}} \widetilde{\epsilon_{jl}} a^{(l)}_{p_l}) = - \widetilde{r^{(l)}_{jp^{l_j}}} \widetilde{\epsilon_{jl}} a^{(l)}_{p_l}$, which shows that there does not exist a global p-basic subgroup B_{1p} of B_1 such that $\overline{(B_{1p})_{p_l}} = (B_{1p} + A^*_{1p_l})/A^*_{1p_l} = \rho^{-1}_{p_l}(B^{(p_l)}_p) = \left(\oplus_{j=1,\dots,n} \langle b_{jp^{l_j}} - \widetilde{r^{(l)}_{jp^{l_j}}} \widetilde{\epsilon_{jl}} a_{p_l} + A^*_{1p_l} \rangle \right) \oplus \langle a_p + A^*_{1p_l} \rangle$.
However, choose $B^{(p_l)}_p = \left(\oplus_{j=1,\dots,n} \langle d^{(l)}_{jp^{l_j}} + \widetilde{r^{(l)}_{jp^{l_j}}} \widetilde{\epsilon_{jl}} a^{(l)}_{p_l} \rangle \right) \oplus \langle \delta_{li_0} a^{(l)}_{p_l} \rangle$ as a p-basic subgroup of $B^{(p_l)}$. Since $h_{1p_l} \left(\nu_j(\frac{1}{p^{l_j}}) \right) = (d^{(l)}_{jp^{l_j}} + \widetilde{r^{(l)}_{jp^{l_j}}} \widetilde{\epsilon_{jl}} a^{(l)}_{p_l}) - g^{(p_l)}_{(1)} \left(\nu_j(\frac{1}{p^{l_j}}) \right) = 0^{(l)}$ for any $l \in \mathbf{N}$, there exists a global p-basic subgroup $B_{1p} = \left(\oplus_{j=1,\dots,n} \langle b_{jp^{l_j}} \rangle \right) \oplus \langle a_p \rangle$ of B_1 such that $\overline{(B_{1p})_{p_l}} = \rho^{-1}_{p_l}(B^{(p_l)}_p) = \left(\oplus_{j=1,\dots,n} \langle b_{jp^{l_j}} + A^*_{1p_l} \rangle \right) \oplus \langle a_p + A^*_{1p_l} \rangle$.

References

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[1] Infinite abelian groups, Vols. I, II, Academic Press, New York-London (1970, 1973).

[2] Abelian p-groups and mixed groups, Sém. Math. Sup., 70, Presse Univ. Montréal (1980).