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# ベクトル空間と CHAIN 完備な半順序ベクトル空間の直積における分離定理について

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The separation theorem is one of the most fundamental theorems in the functional analysis and the optimization theory. Let $X$ be a vector space and $A$ a subset of $X$. We denote ${}^{l}A$ the linear span of $A$ and ${}^{i}A$ denotes the relatively algebraic interior of $A$. If ${}^{l}A = X$, then it coincide the core of $A$. We can obtain the separation theorem in the vector space as follows:

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| Let $X$ be a vector space, $X^*$ its algebraic dual space, $A$, $B$ subsets of $X$ such that relatively algebraic interior ${}^{i}A$ and ${}^{i}B$ are non-empty. Then there exist a $u \in X^*$, $u \neq 0$, and $\lambda \in R$ such that $\langle u, x\rangle \leq \lambda \leq \langle u, y\rangle$ for any $x \in A$ and any $y \in B$, $\langle u, z\rangle \neq \lambda$ for at least one $z \in A \cup B$ if and only if ${}^{i}A \cap {}^{i}B = \emptyset$. |

It is known that this theorem establishes in case where the range space is a Dedekind complete Riesz space. It is known that the separation theorem establishes in the Cartesian product space of a vector space and a Dedekind complete ordered vector space; see [2, 5]. Under certain assumptions, two non-void subset of product space can separated by an affine manifold of that product space. However, when we consider the Cartesian product of a vector space and a chain complete partially ordered vector space, two subsets in that product space are not separated by an affine manifold of that product space.

In this talk, we give a separation type theorem in the Cartesian product of a vector space and a chain complete partially ordered vector space (Theorem 1) using a scalarization method for a vector space. Scalarization methods are one of important methods in optimization theory, there are several applications, see [4].

Let $R$ be the set of real numbers, $N$ the set of natural numbers, $I$ an indexed set, $Y$ an ordered vector space. This is to say that $Y$ is a real vector space endowed with an associated partial order $\leq_K$ induced by a proper convex cone $K (K \neq \emptyset,\ K \neq Y,\ K + K \subset K)$ as follows;

$$x \leq_K y \text{ if } y - x \in K \text{ for } x, y \in Y.$$

It is well known that $\leq_K$ is reflexive antisymmetric and transitive when $K$ is convex. Moreover, $\leq_K$ has invertible properties to vector space structure as translation and scalar multiplication. Let $Z$ be a subset of $Y$. The set $Z$ is called a *chain* if any two elements are comparable, that is, $x \leq_K y$ or $y \leq_K x$ for any $x$, $y \in Z$. $Y$ is said to be *chain complete* if every nonempty chain of $Y$ which is bounded from below has an infimum; $Y$ is said to be *Dedekind complete* if every nonempty subset of $Y$ which is bounded from below has an infimum. A partially ordered vector space is (upward) directed if for any $x$,$y \in Y$ there exists $z \in Y$ such that $x \leq_K z$ and $y \leq_K z$. Let $\overline{R} = R \cup \{\infty\}$ and for the function $\phi : Y \to \overline{R}$, we define the domain and epigraph by

$$\mathrm{D}(\phi) = \{y \in Y \mid \phi(y) < \infty\},\ \mathrm{epi}\,\phi = \{(y, t) \in Y \times R \mid \phi(y) < t\}.$$

$\phi$ is said to be convex if $\mathrm{epi}\,\phi$ is a convex set. $\phi$ is said to be proper if $\mathrm{D}(\phi) \neq \emptyset$ and does not have the value $-\infty$. Let $B \subset Y$. $\phi$ is said to be $B$-monotone if $y_1 - y_2 \in B$ implies $\phi(y_1) \leq_K \phi(y_2)$. We assume that convex cone $K$ contains the rays generated by $k^0 \in Y$, that is,

(1) $$K + [0, \infty) \cdot k^0 \subset K.$$

We move $-K$ along this ray and consider the set

$$K' = \{(y, t) \in Y \times R \mid y \in tk^0 - K\}.$$

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In this case, note that $K'$ is of epigraph type. Since $K'$ is epigraph type, we associated with $K$ and $k^0$ define the function by

$$\phi_{K,k^0}(y) = \inf\{t \in R \mid y \in tk^0 - K\}.$$

Let $K$ be a closed proper convex cone and $k^0 \in Y$. Then we have $\phi_{K,k^0}$ is $K$-monotone if and only if $K + K \subset K$.

Let $X$ be a vector space, $Y$ a chain complete directed partially ordered vector space. Let $f \in \mathcal{L}(X, Y)$, $g \in \mathcal{L}(Y, Y)$, $t_0$ a point in $R$, $K$ a proper closed convex cone in $Y$, $k^0 \in Y \setminus \{0\}$ and $\phi_{K,k^0}$ be a scalarizing function from $Y$ to $R$ which is bounded and proper. Then

$$H = \{(x, y) \in X \times Y \mid \phi_{K,k^0}(f(x) + g(y)) = t_0\}$$

is a subset in $X \times Y$. Let $A$, $B$ be nonempty subsets of $X \times Y$. It is said that $H$ separates $A$ and $B$ if

$$H_- = \{(x, y) \in X \times Y \mid \phi_{K,k^0}(f(x) + g(y)) \leq t_0\} \supset A,$$
$$H_+ = \{(x, y) \in X \times Y \mid \phi_{K,k^0}(f(x) + g(y)) \geq t_0\} \supset B.$$

Let $A$ be a nonempty subsets of $X \times Y$. The operator $P_X$ defined by $P_X(x, y) = x$ for any $(x, y) \in X \times Y$ is called the *projection* of $X \times Y$ onto $X$ and $P_Y$ defined by $P_Y(x, y) = y$ for any $(x, y) \in X \times Y$ is called the *projection* of $X \times Y$ onto $Y$, respectively. Then $P_X \in \mathcal{L}(X \times Y, X)$ and $P_Y \in \mathcal{L}(X \times Y, Y)$, respectively. We define

$$P_X(A) = \{x \in X \mid \text{ there exists } y \in Y \text{ such that } (x, y) \in A\},$$
$$P_Y(A) = \{y \in Y \mid \text{ there exists } x \in X \text{ such that } (x, y) \in A\}.$$

We take chain $C \subset P_Y(A - B)$ and define

$$P_X^C(A - B) = \{x \in X \mid \text{ there exists } y \in C \text{ such that } (x, y) \in A - B\}.$$

A subset $A$ of $X \times Y$ is a cone if $\lambda > 0$ implies $\lambda A \subset A$. The set

$$\mathrm{Cone}(A) = \{\lambda z \in X \times Y \mid \lambda \geq 0, z \in A\}$$

is called the cone span of $A$. If $A$ is convex, then $\mathrm{Cone}(A)$ is convex. We obtain the following separation theorem.

**Theorem 1.** *Let $K$ be a proper closed convex cone in $Y$, $k^0 \in K \backslash \{-K\}$ and $\phi_{K,k^0}$ a scalarizing function from $Y$ to $R$. Let $A$ and $B$ be subsets of $X \times Y$ such that* $\mathrm{Cone}(A - B)$ *is a convex cone, and $C$ be non-empty chain of $P_Y(A - B)$. Assume that the following* (i) *and* (ii) *hold* :
(i) $0 \in {}^iP_X^C(A - B)$ *and* ${}^lP_X^C(A - B) = X$.
(ii) *If $(x, y_1) \in A$ and $(x, y_2) \in B$, then $y_1 \leq_K y_2$ holds.*
*Then there exists an $f \in \mathcal{L}(X, Y)$ and a point $t_0 \in R$ such that $H = \{(x, y) \in X \times Y \mid \phi_{K,k^0}(f(x) - y) = t_0\}$ separates $A$ and $B$.*

## References


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