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The Weierstrass Gap Sets for Quadruples on Compact Riemann Surfaces II

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Abstract: Let M be a compact Riemann surface of genus g. Let $P_i (i = 1, \cdots, 4)$ be 4 distinct points on M. We denote $G(P_1, P_2, P_3, P_4)$ the Weierstrass gap set. We prove that, for large g, the upper bound of $\#G(P_1, P_2, P_3, P_4)$ is attained if and only if M is hyperelliptic and $|2P_i| = g^1_2$.

1. Introduction

Let M be a compact Riemann surface of genus $g \geq 2$, and let $P_1, \cdots, P_n$ be n distinct points on M. The Weierstrass gap set $G(P_1, \cdots, P_n)$ is defined by

$$G(P_1, \cdots, P_n) := \{(\gamma_1, \cdots, \gamma_n) \in \mathbb{N}_0^n \mid \nexists \text{meromorphic function } f \text{ on } M \text{ whose pole divisor } (f)_\infty \text{ is } \gamma_1 P_1 + \cdots + \gamma_n P_n\},$$

where $\mathbb{N}_0$ is the set of non-negative integers. In case $n = 1$, $G(P_1)$ is the set of Weierstrass gaps at P_1, and the cardinality $\#G(P_1)$ is equal to g. But in case $n \geq 2$, $\#G(P_1, \cdots, P_n)$ depends on M and $\{P_1, \cdots, P_n\} (\subset M)$. Concerning $\#G(P_1, \cdots, P_n)$, there is a conjecture presented by Ballico and Kim([1]).

Conjecture Assume g is very large with respect to n. Then we have

$$\#G(P_1, \cdots, P_n) \leq \sum_{0 \leq m \leq n} \binom{n}{m}\binom{g}{m} 2^m - \binom{g+n}{n}, \tag{1}$$

and the equality holds if and only if M is hyperelliptic and $|2P_i| = g^1_2 (i = 1, .., n)$, where $\binom{n}{m}$ is the binomial coefficient.

This conjecture is true in case $n = 2$([6],[2]) and $n = 3$([3]). When M is hyperelliptic, the conjecture is also true for general n([1]).

Here, we prove the conjecture affirmatively in case $n = 4$.

Theorem 1 (Main Theorem)**.** *Assume M is a compact Riemann surface with $g = 11$ or $g \geq 13$. Then*

$$\begin{aligned}\#G(P_1, \cdots, P_4) &\leq \sum_{0 \leq m \leq 4} \binom{4}{m}\binom{g}{m} 2^m - \binom{g+4}{4} \\ &= g(14 + 45g + 22g^2 + 15g^3)/24,\end{aligned} \tag{2}$$

and the equality holds if and only if M is hyperelliptic and $|2P_i| = g^1_2$.

2. Proof of Main Theorem

As mentioned above, this theorem is correct if M is hyperelliptic. Then it suffices to show the following inequality.

Proposition 1. *Let M be a non-hyperelliptic curve of genus $g = 11$ or $g \geq 13$, and let $P_1, \cdots, P_4$ be distinct points on M. Then*

$$\#G(P_1, \cdots, P_4) < \sum_{0 \leq m \leq 4} \binom{4}{m}\binom{g}{m} 2^m - \binom{g+4}{4}. \tag{3}$$

The author has already got the following result by showing (3) in case M is a d-gonal curve with $d \geq 5$.

Proposition 2 ([4])**.** *Let M be a d-gonal curve with $d \geq 2$. That is, d is the smallest number attained by the degree of a non-trivial meromorphic function on M. Moreover we assume $d \neq 3, 4$ and $g \geq 5$.*

Then the inequality (2) *is satisfied, and the equality holds if and only if M is hyperelliptic(i.e., $d = 2$) and $|2P_i| = g^1_2$.*

Therefore Theorem 1 means that the condition $d \neq 3, 4$ can be removed when $g = 11$ or $g \geq 13$. The proof of Theorem 1, in case $d = 3$ or 4, is done by pushing forward the methods in [4].

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Definition 1. *Let n be a positive integer. For an arbitrary curve M of genus $g \geq 2$ and distinct points $P_1, \cdots, P_n \in M$, define*

$$\mathcal{K}(P_1, \cdots, P_n) := \{\Gamma = \gamma_1 P_1 + \cdots + \gamma_n P_n | \\ \Gamma \text{ is a canonical divisor on } M, \gamma_i \geq 0 (i = 1, \cdots, n)\}.$$

In particular, for a hyperelliptic curve M_h of genus g and distinct points $Q_i \in M_h$ with $|2Q_i| = g_2^1 (i = 1, \cdots, n)$, $\mathcal{K}_h$ denotes $\mathcal{K}(Q_1, \cdots, Q_n)$.

Since the canonical series of M_h is $(g-1)g_2^1$, we have

$$\#\mathcal{K}_h = \binom{g+n-2}{n-1}. \tag{4}$$

Moreover the following equality has been proved in [1].

$$\#G(Q_1, \cdots, Q_n) = \sum_{0 \leq m \leq n} \binom{n}{m}\binom{g}{m} 2^m - \binom{g+n}{n}. \tag{5}$$

In case $n = 4$, we also know the following result([4]).

Lemma 1. *Put $\mathcal{K} = \mathcal{K}(P_1, P_2, P_3, P_4)$ for distinct $P_i (i = 1, \cdots, 4)$ on a non-hyperellptic curve M. Points $Q_i (i = 1, \cdots, 4)$ are same as in Definition 1. Then*

$$\#G(Q_1, Q_2, Q_3, Q_4) - \#G(P_1, P_2, P_3, P_4) \geq g^3 + \frac{1}{2}g^2 - \frac{1}{2}g - 9\#\mathcal{K}. \tag{6}$$

Therefore the proof of the inequality (3) is reduced to an estimation of $\#\mathcal{K}$. Actually we prove that the right hand side of (6) is positive if M is d-gonal with $d = 3, 4$ and $g = 11$ or ≥ 13 by using the lemma ([5]) bellow.

Lemma 2. *Assume that M is a d-gonal curve with $d \geq 3$.*
(i) *Assume $d \geq 4$, and let C be a positive integer defined by*

$$C := \begin{cases} \binom{[\frac{2g-2}{3}]+3}{3} + \binom{[\frac{2g-2}{3}]+2}{3} & (\text{if } 3 \nmid 2g-2), \\ \binom{\frac{2g-2}{3}+2}{3} + \binom{\frac{2g-2}{3}+1}{3} & (\text{if } 3 | 2g-2). \end{cases}$$

Here $[r] = \max\{n | n \leq r, n \in \mathbb{Z}\}$. Then

$$\#\mathcal{K}(P_1, P_2, P_3, P_4) \leq C. \tag{7}$$

(ii) *When $d = 3$ and $g \geq 11$,*

$$\#\mathcal{K}(P_1, P_2, P_3, P_4) \leq \binom{[\frac{2g-2}{3}]+3}{3}. \tag{8}$$

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