P-8

連分数とユークリッドのアルゴリズム
Continued fractions and the Euclidean algorithm

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Abstract

In this report, we discuss continued fractions. We show that a good rational approximations to a real number α is given by a continued fraction convergent to α. We give basic properties including results related to periodic continued fractions.

1 Continued Fractions

Definition 1
A continued fraction is an expression of the form

$$q_0 + \cfrac{1}{q_1 + \cfrac{1}{q_2 + \cfrac{1}{q_3 \ddots}}}$$

with either a finite or infinite number of entries q_i.
We denote the above by

$$q_0 + \frac{1}{q_1+}\frac{1}{q_2+}\frac{1}{q_3+}\cdots\frac{1}{q_i} \ \mathit{or}\ [q_0, q_1, \cdots, q_i],$$

which is called a partial quotient of the continued fraction.

Theorem 1
Let $q_0, q_1, \cdots$ be a finite (k+1 element) or infinite sequence of positive integers, with the exception that q_0 can be zero, and let a_n and b_n be given by

$$a_0 = q_0, a_1 = q_0q_1 + 1, a_{n+2} = a_{n+1}q_{n+2} + a_n \ ,$$

$$b_0 = 1, b_1 = q_1, b_{n+2} = b_{n+1}q_{n+2} + b_n \quad (*)$$

where $n \leq k$ in the finite case. If α is a real number greater than 1, then

$$[q_0, \cdots, q_n, \alpha] = \frac{\alpha a_n + a_{n-1}}{\alpha b_n + b_{n-1}} \quad \mathit{provided}\ \ n > 0, \quad (1)$$

$$[q_0, \cdots, q_n] = \frac{a_n}{b_n} \ . \quad (2)$$

Proof
The equatioin (1) holds when $n = 1$ and the continued fraction has three entries. For $n > 1$ we assume (1) holds for all continued fractions with $n+2$ entries, then

$$\begin{aligned}[q_0, \cdots, q_{n+1}, \alpha] &= [q_0, \cdots, q_n, q_{n+1} + \frac{1}{\alpha}] \\ &= \frac{(q_{n+1} + \frac{1}{\alpha})a_n + a_{n-1}}{(q_{n+1} + \frac{1}{\alpha})b_n + b_{n-1}} \\ &= \frac{\alpha a_{n+1} + a_n}{\alpha b_{n+1} + b_n} \quad \text{by}(*),\end{aligned}$$

and (1) follows.
For (2) let $\alpha = q_{n+1}$ in (1) and use $(*)$. ■

Lemma 1
Using the notation above we have, for $n \geq 0$,
(i) $\frac{a_n}{b_n} - \frac{a_{n+1}}{b_{n+1}} = \frac{(-1)^{n+1}}{b_nb_{n+1}}$,
(ii) $(a_n, b_n) = 1$,
(iii) if $n > 0$ then $b_{n+1} > b_n$, and so $b_n \geq n$,
(iv) $\frac{a_0}{b_0} < \frac{a_2}{b_2} < \cdots < \frac{a_{2n}}{b_{2n}} < \cdots < \frac{a_{2n+1}}{b_{2n+1}} < \cdots < \frac{a_1}{b_1}$,
(v) all infinite simple continued fractios converge.

Proof
(i) Using the equations $(*)$ we have $a_0b_1 - a_1b_0 = -1$ and $a_nb_{n+1} - a_{n+1}b_n = -(a_{n-1}b_n - a_nb_{n-1})$, now use induction.
(ii) This follows immediately from (i).
(iii) Use $(*)$ and induction, noting that $q_n \geq 1$ for $n > 0$.
(iv) Substituting q_{n+1} for α in Theorem 1 we see that, as $q_{n+1} \geq 1$, $\frac{a_{n+2}}{b_{n+2}}$ lies between $\frac{a_n}{b_n}$ and $\frac{a_{n+1}}{b_{n+1}}$.
But $\frac{a_0}{b_0} < \frac{a_1}{b_1}$, so $\frac{a_0}{b_0} < \frac{a_2}{b_2} < \frac{a_1}{b_1}$.
The result follows by induction.
(v) By (i) and (iv),$\{\frac{a_n}{b_n}\}$ is a Cauchy sequence and so converges. ■

Lemma 2
Using the above notation, if none of $\alpha, \alpha_1, \cdots, \alpha_{n-1}$ are integers, then

$$\alpha = [q_0, \cdots, q_{n-1}, \alpha_n]. \quad (3)$$

Proof
This follows by induction. ■

Definition 2
Let α be a positive real number. Using the notation above and the equations $(*)$ in Theorem 1 , the rational number $\frac{a_n}{b_n}$ is called the $n-th$ convergent to α provided none of $\alpha, \alpha_1, \cdots, \alpha_n$ belong to $\mathbf{Z}$.

Theorem 2
Suppose α is an irrational number. Using the notation above we have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \alpha, \quad (4)$$

$$\mid \alpha - \frac{a_n}{b_n} \mid < \frac{1}{b_n + b_{n+1}} < \frac{1}{b_n^2}. \quad (5)$$

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Proof
By Lemma 1 (v) $\lim_{n\to\infty}\frac{a_n}{b_n}$ exists. By Theorem 1 (with α_{n+1} for α) and Lemma 2, we see that α lies between $\frac{a_{n-1}}{b_{n-1}}$ and $\frac{a_n}{b_n}$. Both parts of the theorem follow by Lemma 1 (iii) and (iv). ■

Theorem 3
(i) If $[q_0,\cdots,q_n]=[q'_0,\cdots,q'_m]$,$q_n>1$, and $q'_m>1$ then $m=n$ and $q_i=q'_i$ for all $i\le n$.
(ii) If $[q_0,\cdots,q_n,\cdots]=[q'_0,\cdots,q'_n,\cdots]$ then $q_n=q'_n$ for all n.

Proof
(i) We have

$$\begin{aligned}[q_0,\cdots,q_n] &= q_0+\frac{1}{[q_1,\cdots,q_n]}\\ &= q'_0+\frac{1}{[q'_1,\cdots,q'_m]}.\end{aligned}$$

Now, as q_n and q'_m are greater than 1, each fraction is proper, so $q_0=q'_0$, and then $[q_0,\cdots,q_n]=[q'_0,\cdots,q'_m]$. Continue this process. Similarly we get (ii). ■

Theorem 4
The continued fraction representation of α is periodic if and only if α is a quadratic number, that is α satisfies a quadratic polynomial equation with rational coefficients.

Proof
Suppose $\alpha=[q_0,\cdots,q_{k-1},q_k^*,\cdots,q_{k+n-1}^*]$ then we have, by Lemma 2 and Theorem 1,

$$\begin{aligned}\alpha &= [q_0,\cdots,q_{k-1},\alpha_k]\\ &= \frac{\alpha_k a_{k-1}+a_{k-2}}{\alpha_k b_{k-1}+b_{k-2}}\end{aligned}$$

and

$$\begin{aligned}[q_k,\cdots,q_{k+n-1},\alpha_k] &= \alpha_k\\ &= \frac{\alpha_k a_{k+n-2}+a_{k+n-3}}{\alpha_k b_{k+n-2}+b_{k+n-3}}\end{aligned}$$

Combining these we obtain a quadratic equation for α with rational coefficients; that is, α is quadratic.
Conversely suppose α is positive and $\alpha=\alpha_0=\frac{c_0+\sqrt{d}}{e_0}$, where c_0, d, and e_0 are integers, d is not square, $e_0\neq 0$, and $e_0\mid d-c_0^2$. Using the continued fraction representation of α given by Lemma 2 we define c_i and e_i, for $i=1,2,\cdots$, to satisfy

$$\alpha_i=\frac{c_i+\sqrt{d}}{e_i} \qquad (6)$$

by the equations

$$c_{i+1}=q_ie_i-c_i,\quad e_{i+1}=\frac{d-c_{i+1}^2}{e_i}. \qquad (7)$$

It is a simple matter to check (by induction) that c_i and e_i are integers and that, $e_i\neq 0$ and $e_i\mid d-c_i^2$, using the equations

$$\begin{aligned}e_{i+1} &= \frac{d-c_{i+1}^2}{e_i}\\ &= \frac{d-c_i^2}{e_i}+2q_ic_i-q_i^2e_i.\end{aligned}$$

Now (6) follows by introduction using the equation $\alpha_i-q_i=\frac{1}{\alpha_{i+1}}$ and (7).
Further let $\alpha'_i=\frac{c_i-\sqrt{d}}{e_i}$. By Theorem 1 and Lemma 1(ⅲ) and 2 we have, taking conjugates and rewriting singling out α'_i,

$$\alpha'_i=\frac{b_{i-2}}{b_{i-1}}\left(\frac{\alpha'_0-\frac{a_{i-2}}{b_{i-2}}}{\alpha'_0-\frac{a_{i-1}}{b_{i-1}}}\right). \qquad (8)$$

By Theorem 2 the term in the parentheses tends to 1 as i tends to infinity, and so there is an n_0 such that $\alpha'_n<0$ if $n>n_0$. Hence $\alpha_n-\alpha'_n=\frac{2\sqrt{d}}{e_n}>0$, and thus $e_n>0$, if $n>n_0$. Also using (7) we have

$$e_n\le e_ne_{n+1}=d-c_{n+1}^2<d \qquad (9)$$

and

$$c_{n+1}^2<c_{n+1}^2+e_ne_{n+1}=d \qquad (10)$$

Hence if $n>n_0$ there can only be finitely many distinct paris $\{c_n,e_n\}$, and so there is a $k>0$ such that, if $n>n_0$,$q_{n+k}=q_n$. As this implies $q_{n+t}=q_{n+k+t}$ where $t\ge 0$, the result follows. ■

References

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