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$\mathcal{C}$-representations of Mixed Abelian Groups II

Takeshi Yasuda *

First, let $p_1 = 2, p_2 = 3, \ldots$ be a listing of the prime numbers in increasing order, and $\langle a_{p_i m}\rangle$ a cyclic group of order $p_i^{2m+1}(m = 0, 1, 2, \ldots)$. And let A be the direct sum of groups $A_{p_i} = \oplus_{m=0}^{\infty}\langle a_{p_i m}\rangle$, one for each i ($i \in \mathbf{N}$), i.e., $A = \oplus_{i\in\mathbf{N}} A_{p_i}$. Also define C by the direct sum of rational groups $C_j = \langle \frac{1}{p_i^k}$ for all $i, k \in \mathbf{N}\rangle$ of type $\mathbf{t}(C_j) = (\infty, \ldots, \infty, \ldots)$, one for each j ($j = 1, \ldots, n$), i. e., $C = \oplus_{j=1,\ldots,n} C_j$, $C_j = \mathbf{Q}$. Then any element $\frac{m_j}{n_j}$ of C_j can be written in the form $\frac{m_j}{n_j} = s_j + \sum_{1\le i\le I_j} \frac{r_{ji}}{p_i^{\alpha_{ji}}}$, where $n_j = \prod_{1\le i\le I_j} p_i^{\alpha_{ji}} (0 \le \alpha_{ji})$, $m_j, s_j, r_{ji} \in \mathbf{Z}$, $0 \le r_{ji} < p_i^{\alpha_{ji}}, p_i \nmid r_{ji}$ if $r_{ji} \ne 0$.

The aim of our study is to give extensions B, B_χ of A by C such that they are not isomorphic, but $\overline{B_{p_l}} = B/A^*_{p_l}$, $\overline{B_{\chi p_l}} = B_\chi/A^*_{p_l}$ are isomorphic and nonsplitting for any p_l, where $A^*_{p_l} = \oplus_{l\ne i\in\mathbf{N}} A_{p_i}$. Further, we obtain $\mathcal{C}$-representations of B, B_χ to find out their structures.

Now, we shall construct a mixed group B_χ mentioned above. In $B_{p_i} = \prod_{m=0}^{\infty}\langle a_{p_i m}\rangle$, the elements $a_{jp_i 0} = (\chi_i a_{p_i 0}, \epsilon_{j1} p_i a_{p_i 1}, \epsilon_{j2} p_i^2 a_{p_i 2}, \epsilon_{j3} p_i^3 a_{p_i 3}, \ldots)$, $a_{jp_i k} = (0, \ldots, 0, \epsilon_{jk} a_{p_i k}, \epsilon_{jk+1} p_i a_{p_i k+1}, \epsilon_{jk+2} p_i^2 a_{p_i k+2}, \epsilon_{jk+3} p_i^3 a_{p_i k+3}, \ldots)$ ($k \in \mathbf{N}$) are of infinite order and satisfy $p_i \nmid a_{jp_i 0}$, $p_i a_{jp_i 1} - a_{jp_i 0} = -\chi_i a_{p_i 0}$, $p_i a_{jp_i k+1} - a_{jp_i k} = -\epsilon_{jk} a_{p_i k}$, where $\chi_i \in \mathbf{N}, 1 \le \chi_i < p_i; \epsilon_{jk} = 0$ if $k \equiv j \pmod n$ or $\epsilon_{jk} = 1$ if $k \not\equiv j \pmod n$. For $i \ne l$, the equations $p_i^k x = \chi_l a_{p_l 0}$, $p_i^k x = \epsilon_{jm} p_l^m a_{p_l m} (m \in \mathbf{N})$ are uniquely solvable in $\langle \chi_l a_{p_l 0}\rangle$, $\langle \epsilon_{jm} p_l^m a_{p_l m}\rangle$, respectively.

Thus $\prod_{i\in\mathbf{N}} B_{p_i}$ contains unique elements $b_{j0}^{(\chi)}, b_{jp_i^k}^{(\chi)}$ ($i, k \in \mathbf{N}, j = 1, \ldots, n$) so as to satisfy $p_i b_{jp_i}^{(\chi)} = b_{j0}^{(\chi)} - \chi_i a_{p_i 0}, p_i b_{jp_i^{k+1}}^{(\chi)} = b_{jp_i^k}^{(\chi)} - \epsilon_{jk} a_{p_i k}$, where $\chi = (\chi_i)_{i\in\mathbf{N}}$. Put $B_\chi = \langle A, b_{j0}^{(\chi)}, b_{jp_i^k}^{(\chi)}$ for $i, k \in \mathbf{N}$, $j = 1, \ldots, n\rangle$ in $\prod_{i\in\mathbf{N}} B_{p_i}$, then it follows that $\overline{B_{\chi p_l}} = B_\chi/A^*_{p_l} = \langle a_{p_l m} + A^*_{p_l}, b_{j0}^{(\chi)} + (1-\chi_l) a_{p_l 0} + A^*_{p_l}, b_{jp_l^k}^{(\chi)} + A^*_{p_l}, b_{jp_i^k}^{(\chi)} + u_{ik} a_{p_l 0} + A^*_{p_l}$, for $l \ne i$ $i, k \in \mathbf{N}, m = 0, 1, 2, \ldots; j = 1, \ldots, n\rangle$ with relations $p_i^k u_{ik} \equiv (1 - \chi_l) \pmod{p_l}$ if $l \ne i$ for certain $u_{ik} \in \mathbf{Z}, 0 \le u_{ik} < p_l$.

In the case of $\chi_i = 1$ for any $i \in \mathbf{N}$, we have $B = \langle A, b_{j0}, b_{jp_i^k}$ for $i, k \in \mathbf{N}, j = 1, \ldots, n\rangle$, where $p_i b_{jp_i} = b_{j0} - a_{p_i 0}, p_i b_{jp_i^{k+1}} = b_{jp_i^k} - \epsilon_{jk} a_{p_i k}$ ($k \in \mathbf{N}$).

With any elements $a_{p_l m} + A^*_{p_l}, b_{j0} + A^*_{p_l}, b_{jp_l^k} + A^*_{p_l}, b_{jp_i^k} + A^*_{p_l}$ ($l \ne i$, $i, k \in \mathbf{N}, m = 0, 1, 2, \ldots; j = 1, \ldots, n$) of $\overline{B_{p_l}} = B/A^*_{p_l}$, we associate elements $a_{p_l m} + A^*_{p_l}, b_{j0}^{(\chi)} + (1 - \chi_l) a_{p_l 0} + A^*_{p_l}, b_{jp_l^k}^{(\chi)} + A^*_{p_l}, b_{jp_i^k}^{(\chi)} + u_{ik} a_{p_l 0} + A^*_{p_l}$ ($l \ne i$, $i, k \in \mathbf{N}, m = 0, 1, 2, \ldots; j = 1, \ldots, n$) of $\overline{B_{\chi p_l}}$, respectively. This association gives rise to an isomorphism $\varphi_{\chi p_l}$ from $\overline{B_{p_l}}$ onto $\overline{B_{\chi p_l}}$ for any p_l.

Next, define a p_l-mixed group $B^{(p_l)}$ in terms of generators and defining relations as follows : It is generated by elements $a_{p_l m}^{(l)}, d_{j0}^{(l)}, d_{jp_i{}^k}^{(l)}$ ($i, k \in \mathbf{N}, m = 0, 1, 2, \ldots;$ $j = 1, \ldots, n$) such that $p_l^{2m+1} a_{p_l m}^{(l)} = 0^{(l)}$, $p_l d_{jp_l}^{(l)} = d_{j0}^{(l)} - a_{p_l 0}^{(l)}$, $p_l d_{jp_l{}^{k+1}}^{(l)} = d_{jp_l{}^k}^{(l)} - \epsilon_{jk} a_{p_l k}^{(l)}$, $p_i d_{jp_i}^{(l)} = d_{j0}^{(l)}$, $p_i d_{jp_i{}^{k+1}}^{(l)} = d_{jp_i{}^k}^{(l)}$ ($l \ne i$, $i, k \in \mathbf{N}$).
Then $A^{(p_l)} = \langle a_{p_l m}^{(l)}$ for $m = 0, 1, 2, \ldots\rangle$ is the torsion part of $B^{(p_l)}$. And the correspondence

*Tokyo Metropolitan Minamikatsushika High Shool

$\sum_{j=1,\ldots,n} \tau_j\big(s_j + \sum_{1\le i\le I_j} \frac{r_{ji}}{p_i^{\alpha_{ji}}}\big) \longmapsto \sum_{j=1,\ldots,n} \Big\{ s_j\big(d_{j0}^{(l)} + A^{(p_l)}\big) + \sum_{1\le i\le I_j} r_{ji}\big(d_{jp_i{}^{\alpha_{ji}}}^{(l)} + A^{(p_l)}\big)\Big\}$ induces an isomorphism $\kappa_{(p_l)}$ from C onto $B^{(p_l)}/A^{(p_l)}$, where the coordinate injection $\tau_j : \frac{m_j}{n_j} \longmapsto (0,\ldots,0,\frac{m_j}{n_j},0,\ldots,0) \in C$.

With any elements $a_{p_l m}^{(l)}, b_{j0}^{(l)}, b_{jp_i^k}^{(l)}$ $(i,k \in \mathbf{N}, m = 0,1,2,\ldots; j = 1,\ldots,n)$ of $B^{(p_l)}$, we associate elements $a_{p_l m} + A_{p_l}^*, b_{j0} + A_{p_l}^*, b_{jp_i^k} + A_{p_l}^*$, $(i,k \in \mathbf{N}, m = 0,1,2,\ldots; j = 1,\ldots,n)$ of $\overline{B_{p_l}}$, respectively. This association gives rise to an isomorphism ρ_{p_l} from $B^{(p_l)}$ onto $\overline{B_{p_l}}$ for any p_l. Then the composite mapping $\rho_{\chi p_l} = \varphi_{\chi p_l}\rho_{p_l}$ becomes an isomorphism from $B^{(p_l)}$ onto $\overline{B_{\chi p_l}}$ for any p_l.

Thereafter, we consider $\mathcal{C}$-representations of B, B_χ. For $u = \sum_{j=1,\ldots,n} \tau_j\big(s_j + \sum_{1\le i\le I_j} \frac{r_{ji}}{p_i^{\alpha_{ji}}}\big) \in C$, choose a representative $g_\chi(u) = \sum_{j=1,\ldots,n}\big(s_j b_{j0}^{(\chi)} + \sum_{1\le i\le I_j} r_{ji} b_{jp_i^{\alpha_{ji}}}^{(\chi)}\big) \in C$ of the coset $\kappa_\chi(u) = g_\chi(u) + A$. Further put $\overline{g_{\chi p_l}}(u) = g_\chi(u) + A_{p_l}^*$, then the following holds $\rho_{\chi p_l}^{-1}\overline{g_{\chi p_l}}(u) = \sum_{j=1,\ldots,n}\big(s_j d_{j0}^{(l)} + \sum_{1\le i\le I_j} r_{ji} d_{jp_i^{\alpha_{ji}}}^{(l)}\big) - \sum_{j=1,\ldots,n}\big(s_j(1-\chi_l) + \sum_{l\neq i, 1\le i\le I_j} r_{ji} u_{i\alpha_{ji}}\big) a_{p_l 0}^{(l)}$. The function $g_\chi^{(p_l)} = \rho_{\chi p_l}^{-1}\overline{g_{\chi p_l}}$ becomes a representative function from C to $B^{(p_l)}$ relative to $\kappa_{(p_l)}$, which yields the factor set $f_\chi^{(p_l)}$ on C to $A^{(p_l)}$ as follows : $f_\chi^{(p_l)}(u,v) = \sum_{j=1,\ldots,n} f_\chi^{(p_l)}\big(\tau_j\big(\frac{r_{jl'}}{p_l^{\alpha_{jl'}}}\big), \tau_j\big(\frac{r_{jl''}}{p_l^{\alpha_{jl''}}}\big)\big)$ for $u' = \sum_{j=1,\ldots,n} \tau_j\big(s_j' + \sum_{1\le i\le I_j} \frac{r_{ji'}}{p_i^{\alpha_{ji'}}}\big) \in C$ and $u'' = \sum_{j=1,\ldots,n} \tau_j\big(s_j'' + \sum_{1\le i\le I_j} \frac{r_{ji''}}{p_i^{\alpha_{ji''}}}\big) \in C$. We distinguish two cases. Case I : $\alpha_{jl'} \neq \alpha_{jl''}$, i.e., $\alpha_{jl'} > \alpha_{jl''}$ without loss of generality. $f_\chi^{(p_l)}\big(\tau_j\big(\frac{r_{jl'}}{p_l^{\alpha_{jl'}}}\big), \tau_j\big(\frac{r_{jl''}}{p_l^{\alpha_{jl''}}}\big)\big) = -\Delta_{jl}\chi_l a_{p_l 0}^{(l)} - \sum_{k=1}^{\alpha_{jl''}-1} \epsilon_{jk}\Delta_{jl} p_l^k a_{p_l k}^{(l)} - \sum_{k=\alpha_{ji''}}^{\alpha_{ji'}-1} \epsilon_{jk} p_l^{k-\alpha_{ji''}}\big(\Delta_{jl} p_l^{\alpha_{jl''}} - r_{jl''}\big) a_{p_l k}^{(l)}$, where $\Delta_{jl} = \left[\frac{p_l^{\alpha_{jl'}-\alpha_{jl''}} r_{jl''} + r_{jl'}}{p_l^{\alpha_{jl'}}}\right]$ with Gauss' symbol. Case II : $\alpha_{jl} = \alpha_{jl'} = \alpha_{jl''}$. $f_\chi^{(p_l)}\big(\tau_j\big(\frac{r_{jl'}}{p_l^{\alpha_{jl'}}}\big), \tau_j\big(\frac{r_{jl''}}{p_l^{\alpha_{jl''}}}\big)\big) = -\Delta_{jl}\chi_l a_{p_l 0}^{(l)} - \sum_{k=1}^{\alpha_{jl}-\beta_{jl}-1} \epsilon_{jk}\Delta_{jl} p_l^k a_{p_l k}^{(l)} - \sum_{k=\alpha_{jl}-\beta_{jl}}^{\alpha_{jl}-1} \epsilon_{jk} p_l^k\big(\Delta_{jl} + p_l^{\beta_{jl}-\alpha_{jl}} r_{jl}'''\big) a_{p_l k}^{(l)}$, where $\Delta_{jl} = \left[\frac{r_{jl'} + r_{jl''}}{p_l^{\alpha_{jl}}}\right]$, $r_{jl}''' = \frac{r_{jl}}{p_l^{\beta_{jl}}}$ if $p_l^{\beta_{jl}} \| r_{jl} = r_{jl'} + r_{jl''} - \Delta_{jl} p_l^{\alpha_{jl}}$.

Next, put $\mathcal{C} = \big(C,\ \big[(B^{(p_l)}, \kappa_{(p_l)})\big]_{l\in\mathbf{N}}\big)$, $B^{(*)} = \prod_{l\in\mathbf{N}} B^{(p_l)}$, $A^{(*)} = \oplus_{l\in\mathbf{N}} A^{(p_l)}$, and let μ_{p_l} be a coordinate injection from $A^{(p_l)}$ into $A^{(*)}$ acting via $a_{p_l} \mapsto (0,\cdots,0,a_{p_l},0,\cdots)$. In $B^{(*)}$, put $g_\chi^+(u) = \Big(g_\chi^{(p_l)}(u)\Big)_{l\in\mathbf{N}} \in \prod_{l\in\mathbf{N}} B^{(p_l)}$, then we construct a mixed group B_χ^+ generated by adjoining $\big[g_\chi^+(u)\big]_{u\in C}$ to $A^{(*)}$, and write $B_\chi^+ = \Big\langle A^{(*)}, \big[g_\chi^+(u)\big]_{u\in C}\Big\rangle$. Here B_χ^+ is isomorphic to B_χ, and $B_\chi^+ = B\big(\mathcal{C},\ \big[g_\chi^{(p_l)}(u)\big]_{u\in C, l\in\mathbf{N}}\big)$ is a $\mathcal{C}$-representation of B_χ with respect to representative functions. On the other hand, we construct a group B_χ^- as the set of all pairs $\big(u, a^{(*)}\big)_\chi \in C \times A^{(*)}$ with the operation $\big(u', a^{(*)\prime}\big)_\chi + \big(u'', a^{(*)\prime\prime}\big)_\chi = \Big(u' + u'', a^{(*)\prime} + a^{(*)\prime\prime} + f_\chi^{(*)}(u', u'')\Big)_\chi$, where $f_\chi^{(*)}$ is a factor set on C to $A^{(*)}$ as follows : $f_\chi^{(*)}(u', u'') = \sum_{l\in\mathbf{N}} \mu_{p_l} f_\chi^{(p_l)}(u', u'')$. Here B_χ^- is isomorphic to B_χ, and $B_\chi^- = B\big(\mathcal{C},\ \big[f_\chi^{(p_l)}(u', u'')\big]_{u', u''\in C, l\in\mathbf{N}}\big)$ is a $\mathcal{C}$-representation of B_χ with respect to factor sets.

Also, we obtain $\mathcal{C}$-representations $B^+ = B\big(\mathcal{C},\ \big[g^{(p_l)}(u)\big]_{u\in C, l\in\mathbf{N}}\big)$, $B^- = B\big(\mathcal{C},\ \big[f^{(p_l)}(u', u'')\big]_{u', u''\in C, l\in\mathbf{N}}\big)$ of B with respect to representative functions and factor sets, respectively, where $g^{(p_l)}(u) = \sum_{j=1,\ldots,n}\big(s_j d_{j0}^{(l)} + \sum_{1\le i\le I_j} r_{ji} d_{jp_i^{\alpha_{ji}}}^{(l)}\big)$ and $f^{(p_l)}(u', u'')$ is equal to the result of $f_\chi^{(p_l)}(u', u'')$ at the case of $\chi = (1)_{i\in\mathbf{N}}$ mentioned above.

Finally, we wish to verify that B, B_χ are not isomorphic. Because, $h_\chi^{(p_l)}(u) = g^{(p_l)}(u) - g_\chi^{(p_l)}(u) = \sum_{j=1,\ldots,n}\big(s_j(1-\chi_l) + \sum_{l\neq i, 1\le i\le I_j} r_{ji} u_{i\alpha_{ji}}\big) a_{p_l 0}^{(l)}$ implies that $h_\chi^{(p_l)}(u) \neq 0^{(l)}$ for some $u \in C$, except for at most many p_l's.

References

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