C-representations of Mixed Abelian Groups II

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First, let \( p_1 = 2, p_2 = 3, \ldots \) be a listing of the prime numbers in increasing order, and \( \langle a_{p,m} \rangle \) a cyclic group of order \( p_i^{2m+1} \) (\( m = 0, 1, 2, \ldots \)). And let \( A \) be the direct sum of groups \( A_{p_i} = \oplus_{m=0}^{\infty} \langle a_{p,m} \rangle \), one for each \( i \ (i \in \mathbb{N}) \), i.e., \( A = \oplus_{i \in \mathbb{N}} A_{p_i} \). Also define \( C \) by the direct sum of rational groups \( C_j = \langle \frac{1}{p_i^m} \rangle \) for all \( i, k \in \mathbb{N} \) of type \( t(C_j) = (\infty, \ldots, \infty, \ldots) \), one for each \( j \ (j = 1, \ldots, n) \), i.e., \( C = \oplus_{j=1}^{n} C_j \). Then any element \( \frac{m_j}{n_j} \) of \( C_j \) can be written in the form \( \frac{m_j}{n_j} = s_j + \sum_{1 \leq l \leq j} \frac{r_{ji}}{p_i^m} \), where \( n_j = \prod_{1 \leq l \leq j} p_i^{r_{ji}/(0 \leq r_{ji})} \), \( m_j, s_j, r_{ji} \in \mathbb{Z} \).

The aim of our study is to give extensions \( B, B_\chi \) of \( A \) by \( C \) such that they are not isomorphic, but \( B_{p_i} = B/A_{p_i}^\ast \), \( B_{p_i}^\ast = B/A_{p_i}^\ast \) are isomorphic and nonsplitting for any \( p_i \), where \( A_{p_i}^\ast = \oplus_{\nu \neq \nu \in \mathbb{N} \setminus A_{p_i}} \). Further, we obtain \( C \)-representations of \( B, B_\chi \) to find out their structures.

Now, we shall construct a mixed group \( B_{p_i} \) mentioned above. In \( B_{p_i} = \oplus_{m=0}^{\infty} \langle a_{p,m} \rangle \), the elements \( a_{j,p_0} = (\chi_1 a_{p_0,0}, \chi_2 a_{p_0,1}, \chi_3 a_{p_0,2}, \chi_3 a_{p_0,3}, \ldots), a_{j,p,k} = (0, \ldots, 0, \epsilon_{j,k} a_{p,k}, \epsilon_{j,k+1} a_{p,k+1}, \epsilon_{j,k+2} a_{p,k+2}, \epsilon_{j,k+3} a_{p,k+3}, \ldots) \) \((k \in \mathbb{N})\) are of infinite order and satisfy \( p_i \mid a_{j,p_0}, p_i a_{j,p_1} - a_{j,p_0} = -\chi_i a_{p_0,0}; p_i a_{j,p_0} a_{j,p_0} - a_{j,p,k} = -\epsilon_{j,k} a_{p_0} k, \) where \( \chi_i \in \mathbb{N}, 1 \leq \chi_i < p_i; \epsilon_{j,k} = 0 \) if \( k \equiv j \) \((\text{mod} \ n)\) or \( \epsilon_{j,k} = 1 \) if \( k \equiv j \) \((\text{mod} \ n)\). For \( i \neq l \), the equations \( p_i^{l_x} x = \chi_i a_{p_0}, p_i^{j_x} x = \epsilon_{j,m} p_n^{l_x} a_{p,m} (m \in \mathbb{N}) \) are uniquely solvable in \( (\chi_i a_{p_0}), (\epsilon_{j,m} p_n^{l_x} a_{p,m}) \), respectively.

Thus \( \prod_{i \in \mathbb{N}} B_{p_i} \) contains unique elements \( b_{l,j}^{(p)} (i, k \in \mathbb{N}, j = 1, \ldots, n) \) as to satisfy \( p_i b_{l,j}^{(p)} = b_{l,j}^{(p)} - \chi_i a_{p_0,0}; p_i b_{l,j}^{(p)} - b_{l,j}^{(p)} = b_{l,j}^{(p)} - \epsilon_{j,x} a_{p,k}, \) where \( \chi = (\chi_{i})_{i \in \mathbb{N}} \). Put \( B_{\chi} = \langle A, b_{l,j}^{(p)} \rangle \) for \( i, k \in \mathbb{N}, j = 1, \ldots, n \), \( \chi_i = 1 \) for any \( i \in \mathbb{N} \), we have \( B = \langle A, b_{l,j}^{(p)} \rangle \) for \( i, k \in \mathbb{N}, j = 1, \ldots, n \), \( m \in \mathbb{N} \), \( m = 0, 1, 2, \ldots \), \( l \neq 0 \) if \( i \) is not in \( \mathbb{Z} \). 0 \( \leq u_k < p_i \).

In the case of \( \chi_i = 1 \) for any \( i \in \mathbb{N} \), we have \( B = \langle A, b_{l,j}^{(p)}, b_{l,j}^{(p)} \rangle \) for \( i, k \in \mathbb{N}, j = 1, \ldots, n \), \( m \in \mathbb{N} \), \( m = 0, 1, 2, \ldots \), \( l \neq 0 \) if \( i \) is not in \( \mathbb{Z} \). 0 \( \leq u_k < p_i \).

With any \( A_{p_0} + A_{p_1} + A_{p_2} + A_{p_3} + + A_{p_4} + A_{p_5} + A_{p_6} + A_{p_7} + A_{p_8} + \cdots \) \((l \neq 0, i, k \in \mathbb{N}, m = 0, 1, 2, \ldots ; j = 1, \ldots, n) \) of \( B_{\chi} \), respectively.

This association gives rise to an isomorphism \( \varphi_{p_i} \) from \( B_{p_i} \) onto \( B_{p_i} \) for any \( p_i \).

Next, define a \( p_i \)-mixed group \( B^{(p)} \) in terms of generators and defining relations as follows:

It is generated by elements \( a_{p,0}^{(l)}, a_{p,1}^{(l)}, a_{p,2}^{(l)}, a_{p,3}^{(l)} \) \((i, k \in \mathbb{N}, m = 0, 1, 2, \ldots ; j = 1, \ldots, n) \) such that \( p_i^{2m+1} a_{p,0}^{(l)} = 0, p_i^{2m} a_{p,0}^{(l)} = a_{p,0}^{(l)} + a_{p,1}^{(l)}, p_i^{2m} a_{p,2}^{(l)} = a_{p,0}^{(l)} + a_{p,1}^{(l)} - \epsilon_{j,k} a_{p,k}, p_i^{2m} a_{p,2}^{(l)} = d_{p,0}^{(l)}, p_i^{2m} a_{p,2}^{(l)} = d_{p,0}^{(l)} + d_{p,1}^{(l)} + d_{p,2}^{(l)} + d_{p,3}^{(l)} + \cdots \) \((l \neq 0, i, k \in \mathbb{N}) \). Then \( A^{(p)} \) for \( m = 0, 1, 2, \ldots \) is the torsion part of \( B^{(p)} \). And the correspondence
\[
\sum_{j=1,\ldots,n} \tau_j (s_j + \sum_{1 \leq i \leq \text{I}_j} r_{ji} p_{ji}) \mapsto \sum_{j=1,\ldots,n} \left\{ s_j (d^{(l)}_{p_j} + A^{(p_i)}) + \sum_{1 \leq i \leq \text{I}_j} r_{ji} (g^{(l)}_{ij} a^{(i)}_l + A^{(p_i)}) \right\}
\]
induces an isomorphism \( \kappa^{(p_i)} \) from \( C \) onto \( B^{(p_i)}/A^{(p_i)} \), where the coordinate injection \( \tau_j : \frac{m_j}{n_j} \mapsto (0, \ldots, 0, \frac{m_j}{n_j}, 0, \ldots, 0) \in C \).

With any elements \( a_{p_j}^{(l)}(i, k) \in C, m = 0, 1, 2, \ldots; j = 1, \ldots, n \) of \( B^{(p_j)} \), we associate elements \( a_{p_j}^{(l)} + A^{(p_j)}, b_j = 0, \ldots, j = 1, \ldots, n \) of \( B_{p_j} \), respectively. This association gives rise to an isomorphism \( \rho_{p_j} \) from \( B^{(p_j)} \) onto \( B_{p_j} \) for any \( p_j \). Then the composite mapping \( \rho \kappa^{(p_j)} \) becomes an isomorphism from \( B^{(p_j)} \) onto \( B_{p_j} \) for any \( p_j \).

Thereafter, we consider C-representations of \( B, B \). For \( u = \sum_{j=1,\ldots,n} \tau_j (s_j + \sum_{1 \leq i \leq \text{I}_j} r_{ji} p_{ji}) \in C \), choose a representative \( g^{(p_j)}(u) = \sum_{j=1,\ldots,n} \tau_j (s_j + \sum_{1 \leq i \leq \text{I}_j} r_{ji} p_{ji}) \) of \( C \). Further put \( \frac{g^{(p_j)}(u)}{A^{(p_j)}} = g^{(p_j)}(u) + A^{(p_j)} \), then the following holds \( \rho^{(p_j)} \frac{g^{(p_j)}(u)}{A^{(p_j)}} = \sum_{j=1,\ldots,n} \tau_j (s_j + \sum_{1 \leq i \leq \text{I}_j} r_{ji} p_{ji}) - \sum_{j=1,\ldots,n} (s_j (1-\chi_i) + \sum_{1 \neq i, 1 \leq j \leq \text{I}_j} r_{ji} u_{ij} a^{(i)}_l) d^{(l)}_{p_j} \).

The function \( g^{(p_j)} \) becomes a representation of \( C \) relative to \( \kappa^{(p_j)} \), which yields the factor set \( f^{(p_j)} \) on \( C \) to \( A^{(p_j)} \) as follows: \( f^{(p_j)}(u, v) = \sum_{j=1,\ldots,n} \tau_j (s_j + \sum_{1 \leq i \leq \text{I}_j} r_{ji} p_{ji}) \in C \) and \( u^{(p_j)} = \sum_{j=1,\ldots,n} \tau_j (s_j + \sum_{1 \leq i \leq \text{I}_j} r_{ji} p_{ji}) - \sum_{j=1,\ldots,n} (s_j (1-\chi_i) + \sum_{1 \neq i, 1 \leq j \leq \text{I}_j} r_{ji} u_{ij} a^{(i)}_l) d^{(l)}_{p_j} \).

We distinguish two cases. Case I: \( a_{ji} \neq 0, \ldots, \alpha_{ji} \neq 0, \ldots, \alpha_{ji} \neq 0 \), i.e., \( a_{ji} \neq 0, \ldots, \alpha_{ji} \neq 0 \). Then \( f^{(p_j)}(u, v) = -\Delta_j a^{(j)} (\mu, \nu) = -\sum_{k=0}^{a_{ji}^{-1}} e_{jk} \Delta_j a^{(j)} (\mu, \nu) = -\sum_{k=0}^{a_{ji}^{-1}} e_{jk} \Delta_j \).

Next, put \( C = \{ C, \{ [B^{(p_j)}, \kappa^{(p_j)}] \}_{p_j \in \mathbb{N}}, B^{(s)} = \prod_{p_j \in \mathbb{N}} B^{(p_j)}, A^{(s)} = \oplus_{p_j \in \mathbb{N}} A^{(p_j)} \}, \) and let \( \mu_{p_j} \) be a coordinate function from \( A^{(p_j)} \) into \( A^{(s)} \) acting via \( a_{p_j} \mapsto (0, \ldots, 0, a_{p_j}, 0, \ldots, 0) \). In \( A^{(s)} \), put \( g^{(p_j)}(u) = g^{(p_j)}(u) \in \mathbb{N} \).

Here \( B^{(s)} \) is isomorphic to \( B \), and \( B^{(s)} = \{ C, \{ [g^{(p_j)}(u)] \}_{u \in C} \} \) is a \( C \)-representation of \( B \) with respect to representative functions. On the other hand, we construct a group \( B^{(s)} \) as the set of all pairs \( (u, a^{(s)}) \in C \times A^{(s)} \) with the operation \( (u', a^{(s)}) \times (u'', a'^{(s)}) = (u' + u'', a^{(s)} + a'^{(s)} + f^{(s)}(u', u'')) \), where \( f^{(s)} \) is a factor set on \( C \) to \( A^{(s)} \) as follows: \( f^{(s)}(u', u'') = \sum_{p_j \in \mathbb{N}} \mu_{p_j} f^{(p_j)}(u', u'') \). Here \( B^{(s)} \) is isomorphic to \( B \), and \( B^{(s)} = \{ C, \{ f^{(p_j)}(u', u'') \}_{u', u'' \in C} \} \) is a \( C \)-representation of \( B \) with respect to factor sets.