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$\mathcal{C}$-representations of Mixed Abelian Groups IV

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First, let $p_1 = 2, p_2 = 3, \cdots$ be a listing of the prime numbers in increasing order, and $\langle a_{p_i m}\rangle$ a cyclic group of order $p_i^{e_i(m)}(m = 0, 1, 2, \cdots)$, where $e_i(m) = 2\left[\frac{m}{n}\right] + 1$(an integer $n \geq 3$) with Gauss' symbol [ ]. And let A be the direct sum of groups $A_{p_i} = \oplus_{m=0}^{\infty}\langle a_{p_i m}\rangle$, one for each i ($i \in \mathbf{N}$), i.e., $A = \oplus_{i\in\mathbf{N}} A_{p_i}$. Also define C as the direct sum of groups $C_j = \tau_j(\mathbf{Q})$ of type $\mathbf{t}(C_j) = (\infty, \cdots, \infty, \cdots)$, one for each j ($j = 1, \cdots, n$), i.e., $C = \oplus_{j=1}^{n} C_j$, where the coordinate injection $\tau_j : \frac{m_j}{n_j} \longmapsto (0, \cdots, 0, \frac{m_j}{n_j}, 0, \cdots, 0) \in \mathbf{Q} \oplus \cdots \oplus \mathbf{Q}$. Then any element u of C can be uniquely written in the form $u = \sum_{j=1}^{n} \tau_j(\frac{m_j}{n_j}) = \sum_{j=1}^{n} \tau_j(s_j + \sum_{i=1}^{I_j} \frac{r_{ji}}{p_i^{\alpha_{ji}}})$, where $n_j = \prod_{i=1}^{I_j} p_i^{\alpha_{ji}}(\alpha_{ji} \geq 0)$, $m_j, s_j, r_{ji} \in \mathbf{Z}$, $0 \leq r_{ji} < p_i^{\alpha_{ji}}, p_i \nmid r_{ji}$ if $r_{ji} \neq 0$.

The aim of our study is to give extensions $B^{(\alpha)}$ ($\alpha \in \mathbf{N_0} = \mathbf{N}\cup\{0\}$) of A by C such that $B^{(\alpha')}$, $B^{(\alpha'')}$ ($\alpha', \alpha'' \in \mathbf{N_0}$) are not isomorphic, but $\bar{B}_{p_l}^{(\alpha')} = B^{(\alpha')}/A_{p_l}^*$, $\bar{B}_{p_l}^{(\alpha'')} = B^{(\alpha'')}/A_{p_l}^*$ are isomorphic and nonsplitting for any p_l, where $A_{p_l}^* = \oplus_{i\neq l, i\in\mathbf{N}} A_{p_i}$. And we obtain $\mathcal{C}$-representations of $B^{(\alpha)}$ ($\alpha \in \mathbf{N_0}$) to find out their structures.

Now, we shall construct a mixed group $B^{(\alpha)}$ with the properties mentioned above for any $\alpha \in \mathbf{N_0}$. Choose $\chi_{ji}^{(\alpha)}(m)$ ($\alpha, m \in \mathbf{N_0}, i \in \mathbf{N}, j = 1, \cdots, n$) as follows : (I)$\chi_{ji}^{(\alpha)}(0) = 1$ (II)for $0 < m$, $\chi_{ji}^{(0)}(m) = 1$ if $m \not\equiv j \pmod n$; $\chi_{ji}^{(0)}(m) = 0$ if $m \equiv j \pmod n$ (III) (i) for $0 < m < \alpha n$, $\chi_{ji}^{(\alpha)}(m) = 1$ if $m = 1, \cdots, n-3,\quad tn, \cdots, (t+1)n - 3$ $(0 < t < \alpha)$ and $m \not\equiv j \pmod n$; $\chi_{ji}^{(\alpha)}(m) = 0$ if otherwise (ii) for $0 < \alpha n \leq m, \chi_{ji}^{(\alpha)}(m) = \chi_{ji}^{(0)}(m)$. The element $a_{jp_i t}^{(\alpha)} = (0, \cdots, 0, \chi_{ji}^{(\alpha)}(tn)a_{p_i tn}, \cdots, \chi_{ji}^{(\alpha)}((t+1)n-1)a_{p_i(t+1)n-1}, \chi_{ji}^{(\alpha)}((t+1)n)p_i a_{p_i(t+1)n},$ $\cdots, \chi_{ji}^{(\alpha)}((t+2)n-1)p_i a_{p_i(t+2)n-1}, \chi_{ji}^{(\alpha)}((t+2)n)p_i^2 a_{p_i(t+2)n}, \cdots \chi_{ji}^{(\alpha)}((t+3)n-1)p_i^2 a_{p_i(t+3)n-1},$ $\chi_{ji}^{(\alpha)}((t+3)n)p_i^3 a_{p_i(t+3)n}, \cdots) \in B_{p_i} = \prod_{m=0}^{\infty}\langle a_{p_i m}\rangle$ is of infinite order and satisfies $p_i \nmid a_{jp_i 0}^{(\alpha)}$. Also, for $l \neq i$, $l \in \mathbf{N}$, there are elements $x_{jp_l it}^{(\alpha)}$ ($t \in \mathbf{N_0}$) in B_{p_l} such that $p_i^t x_{jp_l it}^{(\alpha)} - a_{jp_l 0}^{(\alpha)} = 0$, where $x_{jp_l i0}^{(\alpha)} = a_{jp_l 0}^{(\alpha)}$. Thus $\prod_{l\in\mathbf{N}} B_{p_l}$ contains unique elements $b_{jp_i^t}^{(\alpha)} = (x_{jp_1 it}^{(\alpha)}, \cdots, x_{jp_{i-1} it}^{(\alpha)}, a_{jp_i t}^{(\alpha)},$ $x_{jp_{i+1} it}^{(\alpha)}, \cdots)$ ($i \in \mathbf{N}, t \in \mathbf{N_0}, j = 1, \cdots, n$) so as to satisfy $p_i b_{jp_i^{t+1}}^{(\alpha)} - b_{jp_i^t}^{(\alpha)} = -\sum_{m=tn}^{(t+1)n-1} \chi_{ji}^{(\alpha)}(m)a_{p_i m}$, where $b_{jp_i^0}^{(\alpha)} = b_{j1}^{(\alpha)}$. And, in $\prod_{l\in\mathbf{N}} B_{p_l}$, let $B^{(\alpha)} = \langle A, b_{jp_i^t}^{(\alpha)}$ for $i \in \mathbf{N}, t \in \mathbf{N_0}, j = 1, \cdots, n\rangle$ be a mixed group obtained by adjoining the elements $b_{jp_i^t}^{(\alpha)}$ ($i \in \mathbf{N}, t \in \mathbf{N_0}, j = 1, \cdots, n$) to A.

Next, define a p_l-mixed group $\widetilde{B}_{p_l}$ in terms of generators and defining relations as follows : it is generated by elements $\tilde{a}_{p_l m}^{[l]}$, $\tilde{b}_{jp_i^t}^{[l]}$ ($i \in \mathbf{N}, m, t \in \mathbf{N_0}, j = 1, \cdots, n$) such that $p_l^{e_l(m)}\tilde{a}_{p_l m}^{[l]} = \tilde{0}^{[l]}$, $p_i \tilde{b}_{jp_i^{t+1}}^{[l]} - \tilde{b}_{jp_i^t}^{[l]} = -\delta_{il}\sum_{m=tn}^{(t+1)n-1} \chi_{jl}^{(0)}(m)\tilde{a}_{p_l m}^{[l]}$ with Kronecker's delta symbol δ_{il}, where $\tilde{b}_{jp_i^0}^{[l]} = \tilde{b}_{j1}^{[l]}$. Then $\widetilde{A}_{p_l} = \oplus_{m=0}^{\infty}\langle \tilde{a}_{p_l m}^{[l]}\rangle$ is the torsion part of $\widetilde{B}_{p_l}$. And the correspondence $\sum_{j=1}^{n} \tau_j(s_j + \sum_{i=1}^{I_j} \frac{r_{ji}}{p_i^{\alpha_{ji}}}) \longmapsto \sum_{j=1}^{n} \left\{ s_j(\tilde{b}_{j1}^{[l]} + \widetilde{A}_{p_l}) + \sum_{i=1}^{I_j} r_{ji}(\tilde{b}_{jp_i^{\alpha_{ji}}}^{[l]} + \widetilde{A}_{p_l}) \right\}$ induces an isomorphism $\widetilde{\kappa}_{p_l}$ from C onto $\widetilde{B}_{p_l}/\widetilde{A}_{p_l}$. Here, denote $b_{jp_l^t}^{(\alpha)\prime} = b_{jp_l^t}^{(\alpha)} + \delta_{<}(t, \alpha)\sum_{s=t+1}^{\alpha}\sum_{m=sn-2}^{sn-1} \chi_{jl}^{(0)}(m)p_l^{s-t-1}a_{p_l m}$,

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$b_{jp_i^t}^{(\alpha)\prime} = b_{jp_i^t}^{(\alpha)} + \sum_{s=1}^{\alpha}\sum_{m=sn-2}^{sn-1} r_{jp_i^t}^{(s)}(m)p_l^{s-1}a_{p_l m}$ for $l \neq i,\ i \in \mathbf{N}$, where $\delta_<(t,\alpha) = 1$ if $t < \alpha$; $\delta_<(t,\alpha) = 0$ if $t \geqq \alpha$, and for $m = sn-2, sn-1 (s = 1,\cdots,\alpha), r_{jp_i^t}^{(s)}(m) \in \mathbf{Z}$ such that $p_i^t r_{jp_i^t}^{(s)}(m) \equiv \chi_{jl}^{(0)}(m) \pmod{p_l^s}$.

And with any elements $\tilde{a}_{p_l m}^{[l]},\ \tilde{b}_{jp_i^t}^{[l]} (i \in \mathbf{N}, m, t \in \mathbf{N_0}, j = 1,\cdots,n\ )$ of $\widetilde{B}_{p_l}$, we associate elements $a_{p_l m} + A_{p_l}^*, b_{jp_i^t}^{(\alpha)\prime} + A_{p_l}^*$ of $\bar{B}_{p_l}^{(\alpha)} (i \in \mathbf{N}, m, t \in \mathbf{N_0}, j = 1,\cdots,n\ )$, respectively. This association gives rise to an isomorphism $\rho_{p_l}^{(\alpha)}$ from $\widetilde{B}_{p_l}$ onto $\bar{B}_{p_l}^{(\alpha)}$ for any p_l.

Thereafter, we consider $\mathcal{C}$-representations of $B^{(\alpha)}$ $(\alpha \in \mathbf{N_0})$. For $u = \sum_{j=1}^n \tau_j(s_j + \sum_{i=1}^{I_j} \frac{r_{ji}}{p_i^{\alpha_{ji}}}) \in C$, choose a representative $g^{(\alpha)}(u) = \sum_{j=1}^n (s_j b_{j1}^{(\alpha)} + \sum_{i=1}^{I_j} r_{ji} b_{p_i^{\alpha_{ji}}}^{(\alpha)}) \in C$ of the coset $\kappa^{(\alpha)}(u) = \sum_{j=1}^n \{s_j(b_{j1}^{(\alpha)} + A) + \sum_{i=1}^{I_j} r_{ji}(b_{p_i^{\alpha_{ji}}}^{(\alpha)} + A)\}$. Further put $\bar{g}_{p_l}^{(\alpha)}(u) = g^{(\alpha)}(u) + A_{p_l}^*$, then the following holds

$$(\rho_{p_l}^{(\alpha)})^{-1}\bar{g}_{p_l}^{(\alpha)}(u) = \sum_{j=1}^n (s_j \tilde{b}_{j1}^{[l]} + \sum_{i=1}^{I_j} r_{ji}\tilde{b}_{jp_i^{\alpha_{ji}}}^{[l]}) - \delta_<(0,\alpha)\sum_{j=1}^n \lfloor s_j \sum_{s=1}^{\alpha}\sum_{m=sn-2}^{sn-1}\chi_{jl}^{(0)}(m)p_l^{s-1}\tilde{a}_{p_l m}^{[l]} + \sum_{i=1}^{I_j}\{\delta_{il} r_{ji}\delta_<(\alpha_{ji},\alpha)\sum_{s=\alpha_{ji}+1}^{\alpha}\sum_{m=sn-2}^{sn-1}\chi_{jl}^{(0)}(m)p_l^{s-\alpha_{ji}-1}\tilde{a}_{p_l m}^{[l]} + (1-\delta_{il})r_{ji}\sum_{s=1}^{\alpha}\sum_{m=sn-2}^{sn-1} r_{jp_i^{\alpha_{ji}}}^{(s)}(m)p_l^{s-1}\tilde{a}_{p_l m}^{[l]}\}\rfloor.$$

The $\tilde{g}_{p_l}^{(\alpha)} = (\rho_{p_l}^{(\alpha)})^{-1}\bar{g}_{p_l}^{(\alpha)}$ becomes a representative function from C to $\widetilde{B}_{p_l}$ relative to $\widetilde{\kappa}_{p_l}$, which yields the factor set $\tilde{f}_{p_l}^{(\alpha)}$ on C to $\widetilde{A}_{p_l}$ as follows : $\tilde{f}_{p_l}^{(\alpha)}(u', u'') = \sum_{j=1}^n \sum_{i=1}^{I_j} \tilde{f}_{p_l}^{(\alpha)}\big(\tau_j\big(\frac{r_{ji'}}{p_i^{\alpha_{ji'}}}\big), \tau_j\big(\frac{r_{ji''}}{p_i^{\alpha_{ji''}}}\big)\big)$ for $u' = \sum_{j=1}^n \tau_j(s_j' + \sum_{i=1}^{I_j}\frac{r_{ji'}}{p_i^{\alpha_{ji'}}}) \in C$ and $u'' = \sum_{j=1}^n \tau_j(s_j'' + \sum_{i=1}^{I_j}\frac{r_{ji''}}{p_i^{\alpha_{ji''}}}) \in C$.

We distinguish two cases. Case I : for $i \neq l,\ i \in \mathbf{N}$ $\tilde{f}_{p_l}^{(\alpha)}\big(\tau_j\big(\frac{r_{ji'}}{p_i^{\alpha_{ji'}}}\big), \tau_j\big(\frac{r_{ji''}}{p_i^{\alpha_{ji''}}}\big)\big) = \tilde{0}^{[l]}$. Case II : for $i = l$ (1) $\alpha_{jl'} \neq \alpha_{jl''}$, i.e., $\alpha_{jl'} < \alpha_{jl''}$ without loss of generality. $\tilde{f}_{p_l}^{(\alpha)}\big(\tau_j\big(\frac{r_{jl'}}{p_l^{\alpha_{jl'}}}\big), \tau_j\big(\frac{r_{jl''}}{p_l^{\alpha_{jl''}}}\big)\big) = -\sum_{s=0}^{\alpha_{jl''}-1}\sum_{m=sn}^{(s+1)n-1}\chi_{jl}^{(\alpha)}(m)\Delta_{jl}p_l^s\tilde{a}_{p_l m}^{[l]} + \sum_{s=\alpha_{jl'}}^{\alpha_{jl''}-1}\sum_{m=sn}^{(s+1)n-1}\chi_{jl}^{(\alpha)}(m)r_{jl}' p_l^{s-\alpha_{jl'}}\tilde{a}_{p_l m}^{[l]}$, where $\Delta_{jl} = \left[\frac{r_{jl'}p_l^{\alpha_{jl''}-\alpha_{jl'}} + r_{jl''}}{p_l^{\alpha_{jl''}}}\right]$ with Gauss' symbol. (2) $\alpha_{jl} = \alpha_{jl'} = \alpha_{jl''}$. $\tilde{f}_{p_l}^{(\alpha)}\big(\tau_j\big(\frac{r_{jl'}}{p_l^{\alpha_{jl}}}\big), \tau_j\big(\frac{r_{jl''}}{p_l^{\alpha_{jl}}}\big)\big) = -\sum_{s=0}^{\alpha_{jl}-1}\sum_{m=sn}^{(s+1)n-1}\chi_{jl}^{(\alpha)}(m)\Delta_{jl}p_l^s\tilde{a}_{p_l m}^{[l]} - \delta_<(0,\beta_{jl})\sum_{s=\alpha_{jl}-\beta_{jl}}^{\alpha_{jl}-1}\sum_{m=sn}^{(s+1)n-1}\chi_{jl}^{(\alpha)}(m)r_{jl}''' p_l^{s-(\alpha_{jl}-\beta_{jl})}\tilde{a}_{p_l m}^{[l]}$, where $\Delta_{jl} = \left[\frac{r_{jl'} + r_{jl''}}{p_l^{\alpha_{jl}}}\right]$, $r_{jl}''' = \frac{r_{jl}}{p_l^{\beta_{jl}}}$ if $p_l^{\beta_{jl}} \| r_{jl} = r_{jl'} + r_{jl''} - \Delta_{jl}p_l^{\alpha_{jl}}$.

Next, put $\mathcal{C} = \big(C,\ \big[(\widetilde{B}_{p_l}, \widetilde{\kappa}_{p_l})\big]_{l \in \mathbf{N}}\big)$, $A^{(*)} = \oplus_{l \in \mathbf{N}}\widetilde{A}_{p_l}$, and $g^{(\alpha)+}(u) = \big(\tilde{g}_{p_l}^{(\alpha)}(u)\big)_{l \in \mathbf{N}} \in \prod_{l \in \mathbf{N}}\widetilde{B}_{p_l}$. And, in $\prod_{l \in \mathbf{N}}\widetilde{B}_{p_l}$, let $B^{(\alpha)+}$ be a mixed group obtained by adjoining $\big[g^{(\alpha)+}(u)\big]_{u \in C}$ to $A^{(*)}$, which is isomorphic to $B^{(\alpha)}$ for any $\alpha \in \mathbf{N_0}$. Therefore, we simply represent $B^{(\alpha)+}$ by $B\big(\mathcal{C},\ \big[\tilde{g}_{p_l}^{(\alpha)}(u)\big]_{u \in C, l \in \mathbf{N}}\big)$ and call it a $\mathcal{C}$-representation of $B^{(\alpha)}$ with respect to representative functions.

On the other hand, we construct a mixed group $B^{(\alpha)-}$ as the set of all pairs $\big(u, a^{(*)}\big) \in C \times A^{(*)}$ with the operation $\big(u', a^{(*)\prime}\big) + \big(u'', a^{(*)\prime\prime}\big) = \big(u' + u'', a^{(*)\prime} + a^{(*)\prime\prime} + f^{(\alpha)-}(u', u'')\big)$, where $f^{(\alpha)-}$ is a factor set on C to $A^{(*)}$ as follows : $f^{(\alpha)-}(u', u'') = \sum_{l \in \mathbf{N}}\mu_{p_l}\tilde{f}_{p_l}(u', u'')$ with a coordinate injection μ_{p_l} from $\widetilde{A}_{p_l}$ into $A^{(*)}$. Here $B^{(\alpha)-}$ is isomorphic to $B^{(\alpha)}$, and $B^{(\alpha)-} = B\big(\mathcal{C},\ \big[\tilde{f}_{p_l}(u', u'')\big]_{u', u'' \in C, l \in \mathbf{N}}\big)$ is a $\mathcal{C}$-representation of $B^{(\alpha)}$ with respect to factor sets.

References

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